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Laplace operators in deRham complexes associated with measures on configuration spaces

Sergio Albeverio [a,b,c,d,e], Alexei Daletskii [b,d,f,g], Yuri Kondratiev [b,d,g,h], Eugene Lytvynov [a,b,d,*]

[a] *Institut für Angewandte Mathematik, Abteilung fur Stochastik, Universität Bonn, Wegelerstrasse 6, D-53115 Bonn, Germany*
[b] *SFB 256, Univ. Bonn, Bonn, Germany*
[c] *SFB 237, Bochum, Düsseldorf, Essen, Germany*
[d] *BiBoS, Univ. Bielefeld, Bielefeld, Germany*
[e] *CERFIM, Acc. Arch. (USI), Locarno, Switzerland*
[f] *Nottingham Trent University, Burton Street, Nottingham NG1 4BU, UK*
[g] *Institute of Mathematics, Kiev, Ukraine*
[h] *Fakultät für Mathematik, Universität Bielefeld, Postfach 10 01 31, D-33501 Bielefeld, Germany*

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Abstract

Let Γ_X denote the space of all locally finite configurations in a complete, stochastically complete, connected, oriented Riemannian manifold X, whose volume measure m is infinite. In this paper, we construct and study spaces $L^2_\mu\Omega^n$ of differential n-forms over Γ_X that are square integrable with respect to a probability measure μ on Γ_X. The measure μ is supposed to satisfy the condition Σ'_m (generalized Mecke identity) well known in the theory of point processes. On $L^2_\mu\Omega^n$, we introduce bilinear forms of Bochner and deRham type. We prove their closabilty and call the generators of the corresponding closures the Bochner and deRham Laplacian, respectively. We prove that both operators contain in their domain the set of all smooth local forms. We show that, under a rather general assumption on the measure μ, the space of all Bochner-harmonic μ-square-integrable forms on Γ_X consists only of the zero form. Finally, a Weitzenböck type formula connecting the Bochner and deRham Laplacians is obtained. As examples, we consider (mixed) Poisson measures, Ruelle

* Corresponding author. Present address: Institut für Angewandte Mathematik, Abteilung für Stochastik, Universität Bonn, Wegelerstrasse 6, Bonn D-53115, Germany. Tel.: +49-228-73-38-45; fax: +49-228-73-95-37.
E-mail addresses: albeverio@uni-bonn.de (S. Albeverio), alexei.daletskii@ntu.ac.uk (A. Daletskii), kondrat@mathematik.uni-bielefeld.de (Y. Kondratiev), lytvynov@wiener.iam.uni-bonn.de (E. Lytvynov).

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type measures on $\Gamma_{\mathbb{R}^d}$, and Gibbs measures in the low activity–high temperature regime, as well as Gibbs measures with a positive interaction potential on Γ_X.

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1. Introduction

Let Γ_X denote the space of all locally finite configurations in a complete, stochastically complete, connected, oriented Riemannian manifold X of infinite volume. The growing interest in geometry and analysis on the configuration spaces Γ_X can be explained by the fact that these naturally appear in different problems of statistical mechanics, quantum physics, and the theory of point processes. In [7–9], an approach to the configuration spaces as infinite-dimensional manifolds was initiated. This approach was motivated by the theory of representations of diffeomorphism groups (see [27,28,53]; these references as well as [9,11] also contain discussion of relations with quantum physics). We refer the reader to [10,11,38,50], and references therein for further discussion of analysis on the configuration spaces and applications. Let us stress that Γ_X is essentially the space of infinite configurations. Geometry and topology of the spaces of finite configurations have been discussed by many authors, see [25] and the references therein, and form quite a different field.

On the other hand, stochastic differential geometry of infinite-dimensional manifolds, in particular, their (stochastic) cohomologies and related questions (Laplace operators and Sobolev calculus in spaces of differential forms, harmonic forms, Hodge decomposition), has been a very active topic of research in recent years. It turns out that many important examples of infinite-dimensional nonflat spaces (loop spaces, product manifolds, configuration spaces) are naturally equipped with probability measures (Brownian bridge, Poisson measures, Gibbs measures). Properties of these measures depend in a nontrivial way on the differential geometry of the underlying spaces themselves, and play therefore a significant role in their study. Moreover, in many cases the absence of a proper smooth manifold structure makes it more natural to work with L^2-objects (such as functions, sections, etc.) on these infinite-dimensional spaces, rather than to define analogs of the smooth ones.

Thus, the concept of an L^2-deRham complex has an important meaning in this framework. The study of L^2-cohomologies for finite-dimensional manifolds, initiated in [16], has been a subject of many works (see, e.g. [18,22,24] and the review papers [41,46]). In the infinite-dimensional case, loop spaces have been most studied [23,29,36,37], the papers [23,37] containing also a review of the subject. The deRham complex on infinite product manifolds with Gibbs measures (which appear in connection with problems of classical statistical mechanics) was constructed in [1,2] (see also [17] for the case of the infinite-dimensional torus). We should also mention the papers [6,13–15,52], where the case of a flat (Hilbert) state space has been considered (the L^2-cohomological structure turns out to be nontrivial even in this case due to the existence of interesting measures on such a space).

In [3,4], the authors started the study of differential forms over the infinite-dimensional space Γ_X and the corresponding Laplacians (of Bochner and deRham type) acting in the L^2-spaces with respect to a Poisson measure. In [5], the associated L^2-cohomologies have been investigated.

Another approach to the construction of differential forms and related objects over Poisson spaces, based on the "transfer principle" from Wiener spaces, was proposed in [49] (see also [47,48]).

It should be stressed that the choice of an underlying measure plays a crucial role in all these studies. The results of [3–5] have only covered the case of Poisson measures, which are related to mathematical models of "free" systems, i.e., systems without interaction. The choice of more complicated measures, such as Gibbs type perturbations of Poisson measures, is particularly motivated by the study of interacting systems of classical statistical mechanics. Properties of the corresponding Laplace operators may then strongly depend on the choice of an appropriate measure.

In order to develop a reasonable theory covering also this case, we need to restrict ourselves to a class of measures on Γ_X that possess a certain regularity. So, we consider those measures μ which satisfy the following condition: for any measurable function $F : \Gamma_X \times X \to \mathbb{R}$, $F \geq 0$:

$$\int_{\Gamma_X} \mu(\mathrm{d}\gamma) \sum_{x\in\gamma} F(\gamma, x) = \int_{\Gamma_X} \mu(\mathrm{d}\gamma) \int_X \sigma(\gamma, \mathrm{d}x) F(\gamma \cup \{x\}, x), \tag{1.1}$$

where $\sigma(\gamma, \cdot)$ is a Borel measure on X which is absolutely continuous with respect to the volume measure m on X for μ-a.e. $\gamma \in \Gamma_X$. In particular, the Poisson measure with intensity $\rho(x)m(\mathrm{d}x)$ satisfies (1.1) with $\sigma(\gamma, \mathrm{d}x) = \rho(x)m(\mathrm{d}x)$, and in this case (1.1) becomes the classical Mecke identity [43] (see also [30,31]). Furthermore, as shown by Georgii [26] and Nguyen and Zessin [45], (1.1) holds for all Gibbs measures. The class of all probability measures on Γ_X satisfying (1.1) was singled out in [42] (see also [54]), where (1.1) was called condition Σ'_m. A relation between this condition and an integration by parts formula for a measure μ was studied in [38].

An iterated application of (1.1) to a function $F : \Gamma_X \times X^k \to \mathbb{R}$, $k \in \mathbb{N}$, gives rise to a family of random measures $\sigma^{(k)}(\gamma)$ on X^k.

The structure of the present paper is as follows. In Section 2 we recall the definition of a differential form over Γ_X, first given in [3,4], and introduce the spaces $L^2_\mu \Omega^n$ of forms that are square integrable with respect to μ. We construct a unitary isomorhism:

$$I^n : L^2_\mu \Omega^n \to \bigoplus_{k=1}^{n} L^2_\mu \left(\Gamma_X \to \bigcup_{\gamma\in\Gamma_X} L^2_{\sigma^{(k)}(\gamma)} \Psi^n_{\mathrm{sym}}(X^k) \right), \tag{1.2}$$

where $L^2_\mu(\Gamma_X \to \cup_{\gamma\in\Gamma_X} L^2_{\sigma^{(k)}(\gamma)} \Psi^n_{\mathrm{sym}}(X^k))$ is the space of μ-square-integrable mappings:

$$\Gamma_X \ni \gamma \mapsto W(\gamma) \in L^2_{\sigma^{(k)}(\gamma)} \Psi^n_{\mathrm{sym}}(X^k) \tag{1.3}$$

and $L^2_{\sigma^{(k)}(\gamma)}\Psi^n_{\mathrm{sym}}(X^k)$ is a space of n-forms over X^k that are square integrable with respect to $\sigma^{(k)}(\gamma)$ and satisfy some additional conditions. In the case where μ is a Poisson measure π, the isomorphism I^n was constructed in [5].

In Section 3, we define Bochner type operators in $L^2_\mu\Omega^n$. First, we introduce the bilinear form:

$$\mathcal{E}^{\mathrm{B}}_{\mu,n}(W^{(1)}, W^{(2)}) := \int_{\Gamma_X} \langle \nabla^\Gamma W^{(1)}(\gamma), \nabla^\Gamma W^{(2)}(\gamma)\rangle \mu(\mathrm{d}\gamma)$$

on the space of smooth local forms, where ∇^Γ is the covariant derivative on Γ_X (introduced in [3,4]), and prove its closability. We call the corresponding generator $\mathbf{H}^{\mathrm{B}}_{\mu,n}$ the Bochner Laplacian on Γ_X associated with μ.

Further, we show that, under the action of the isomorphism I^n, the form $\mathcal{E}^{\mathrm{B}}_{\mu,n}$ can be expressed via Bochner type bilinear forms $\mathcal{E}^{\mathrm{B}}_{\sigma^{(k)}(\gamma)}$ associated with the measures $\sigma^{(k)}(\gamma)$ on X^k, $k = 1, \dots, n$, μ-a.e. $\gamma \in \Gamma_X$. As an application of this result, we derive sufficient conditions for the space of all Bochner-harmonic μ-square-integrable forms on Γ_X to consist only of the zero form. Let us remark that we do not assume extremality of μ, so that nonconstant μ-square-integrable harmonic functions on Γ_X may in general exist [10].

In Section 4, we introduce and study the structure of the deRham complex in the spaces $L^2_\mu\Omega^n$. Following [5], we first define a Hodge–deRham differential $\mathbf{d}_n$ on the space of smooth local forms. We prove the closability of the $\mathbf{d}_n$'s as operators from $L^2_\mu\Omega^n$ into $L^2_\mu\Omega^{n+1}$ and consider the Hilbert complex:

$$\cdots \overset{\bar{\mathbf{d}}_{n-1}}{\rightarrow} L^2_\mu\Omega^n \overset{\bar{\mathbf{d}}_n}{\rightarrow} L^2_\mu\Omega^{n+1} \overset{\bar{\mathbf{d}}_{n+1}}{\rightarrow} \cdots,$$

where $\bar{\mathbf{d}}_n$'s are the corresponding closures. Next, we define a Hodge–deRham Laplacian $\mathbf{H}^{\mathrm{R}}_{\mu,n}$ as the generator of the closed form:

$$\mathcal{E}^{\mathrm{R}}_{\mu,n}(W^{(1)}, W^{(2)}) := (\bar{\mathbf{d}}_n W^{(1)}, \bar{\mathbf{d}}_n W^{(2)})_{L^2_\mu\Omega^{n+1}} + (\mathbf{d}^*_{n-1} W^{(1)}, \mathbf{d}^*_{n-1} W^{(2)})_{L^2_\mu\Omega^{n-1}}$$

on $L^2_\mu\Omega^n$ with domain $D(\mathcal{E}^{\mathrm{R}}_{\mu,n}) = D(\bar{\mathbf{d}}_n) \cap D(\mathbf{d}^*_{n-1})$. We prove that, under certain additional conditions on μ, the domain of the operator $\mathbf{H}^{\mathrm{R}}_{\mu,n}$ contains smooth local forms. This gives us a possibility to prove, for $\mathbf{H}^{\mathrm{B}}_{\mu,n}$ and $\mathbf{H}^{\mathrm{R}}_{\mu,n}$, an analog of the Weitzeböck formula.

In Section 4, we consider our main examples: Gibbs measures with pair interaction on Γ_X. More exactly, we consider in details Ruelle type measures on $\Gamma_{\mathbb{R}^d}$ (cf. [51]), and Gibbs measures in the low activity–high temperature regime, as well as Gibbs measures with positive potentials on Γ_X. In these cases, we get more explicit expressions for the Bochner and deRham Laplacians.

2. Differential forms over a configuration space

Let X be a complete, connected, oriented, C^∞ Riemannian manifold of infinite volume. Let d denote the dimension of X. Let $\langle\cdot,\cdot\rangle_x$ denote the inner product in the tangent space

T_xX to X at a point $x \in X$. The associated norm will be denoted by $|\cdot|_x$. Let ∇^X stand for the gradient on X.

The configuration space Γ_X over X is defined as the set of all locally finite subsets (configurations) in X:

$$\Gamma_X := \{\gamma \subset X \,|\, |\gamma_\Lambda| < \infty \text{ for each compact } \Lambda \subset X\}.$$

Here, $\gamma_\Lambda := \gamma \cap \Lambda$ and $|A|$ denotes the cardinality of a set A.

We can identify any $\gamma \in \Gamma_X$ with the positive, integer-valued Radon measure:

$$\sum_{x \in \gamma} \varepsilon_x \in \mathcal{M}(X),$$

where ε_x is the Dirac measure with mass at x, $\sum_{x \in \emptyset} \varepsilon_x :=$zero measure, and $\mathcal{M}(X)$ denotes the set of all positive Radon measures on the Borel σ-algebra $\mathcal{B}(X)$. The space Γ_X is endowed with the relative topology as a subset of the space $\mathcal{M}(X)$ with the vague topology, i.e., the weakest topology on Γ_X with respect to which all maps:

$$\Gamma_X \ni \gamma \mapsto \langle f, \gamma \rangle := \int_X f(x)\gamma(\mathrm{d}x) \equiv \sum_{x \in \gamma} f(x)$$

are continuous. Here, $f \in C_0(X)$ (:= the set of all continuous functions on X with compact support). Let $\mathcal{B}(\Gamma_X)$ denote the corresponding Borel σ-algebra.

The tangent space to Γ_X at a point γ is defined as the Hilbert space:

$$T_\gamma \Gamma_X := L^2(X \to TX; \gamma) \equiv \oplus_{x \in \gamma} T_x X. \tag{2.1}$$

The scalar product and the norm in $T_\gamma\Gamma_X$ will be denoted by $\langle \cdot, \cdot \rangle_\gamma$ and $\|\cdot\|_\gamma$, respectively. Thus, each $V(\gamma) \in T_\gamma\Gamma_X$ has the form $V(\gamma) = (V(\gamma, x))_{x \in \gamma}$, where $V(\gamma, x) \in T_xX$, and

$$\|V(\gamma)\|_\gamma^2 = \sum_{x \in \gamma} |V(\gamma, x)|_x^2.$$

We now recall how to define derivatives of a function $F : \Gamma_X \to \mathbb{R}$. Let $\gamma \in \Gamma_X$ and $x \in \gamma$. By $\mathcal{O}_{\gamma,x}$ we denote an arbitrary open neighborhood of x in X such that $\mathcal{O}_{\gamma,x} \cap (\gamma \setminus \{x\}) = \emptyset$. We define the function

$$\mathcal{O}_{\gamma,x} \ni y \mapsto F_x(\gamma, y) := F(\gamma - \varepsilon_x + \varepsilon_y) \in \mathbb{R}.$$

We say that F is differentiable at $\gamma \in \Gamma_X$ if, for each $x \in \gamma$, the function $F_x(\gamma, \cdot)$ is differentiable at x and

$$\nabla^\Gamma F(\gamma) := (\nabla_x^X F(\gamma))_{x \in \gamma} \in T_\gamma \Gamma_X, \qquad \nabla_x^X F(\gamma) := \nabla^X F_x(\gamma, x).$$

Analogously, the higher order derivatives of F are defined, $(\nabla^\Gamma)^{(k)} F(\gamma) \in (T_\gamma\Gamma_X)^{\otimes k}$, $k \in \mathbb{N}$.

Let $\mathcal{O}_{\mathrm{c}}(X)$ denote the set of all open, relatively compact sets in X. A function $F : \Gamma_X \to \mathbb{R}$ is called local if there exists $\Lambda \in \mathcal{O}_{\mathrm{c}}(X)$ such that $F(\gamma) = F(\gamma_\Lambda)$ for each $\gamma \in \Gamma_X$.

Any function of the form:

$$F(\gamma) = g_F(\langle\varphi_1, \gamma\rangle, \dots, \langle\varphi_N, \gamma\rangle), \tag{2.2}$$

where $g_F \in C_{\mathrm{b}}^\infty(\mathbb{R}^N)$ and $\varphi_1, \dots, \varphi_N \in \mathcal{D} := C_0^\infty(X)$ (:= the set of all infinitely differentiable functions on X with compact support), is local, bounded, infinitely differentiable, and the derivatives of F are polynomially bounded:

$$\forall\, k \in \mathbb{N}\, \exists \varphi \in C_0(X), \quad \varphi \geq 0 : \|(\nabla^\Gamma)^{(k)} F(\gamma)\|^2_{(T_\gamma \Gamma_X)^{\otimes k}} \leq \langle\varphi, \gamma\rangle^k \quad \text{for all } \gamma \in \Gamma_X. \tag{2.3}$$

The set of all functions of the form (2.2) will be denoted by $\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D}, \Gamma_X)$.

Vector fields and first order differential forms on Γ_X will be identified with sections of the bundle $T\Gamma_X$. Higher order differential forms will be identified with sections of the tensor bundles $\wedge^n(T\Gamma_X)$ with fibers:

$$\wedge^n(T_\gamma \Gamma_X) = \wedge^n \left(\oplus_{x\in\gamma} T_x X\right), \tag{2.4}$$

where $\wedge^n(\mathcal{H})$ (or $\mathcal{H}^{\wedge n}$) stands for the nth antisymmetric tensor power of a Hilbert space $\mathcal{H}$. Thus, under a differential form W of order n, $n \in \mathbb{N}$, over Γ_X, we will understand a mapping:

$$\Gamma_X \ni \gamma \mapsto W(\gamma) \in \wedge^n(T_\gamma \Gamma_X). \tag{2.5}$$

We will now recall how to introduce a covariant derivative of a differential form (2.5).

Let $\gamma \in \Gamma_X$ and $x \in \gamma$. We define the mapping

$$\mathcal{O}_{\gamma,x} \ni y \mapsto W_x(\gamma, y) := W(\gamma_y) \in \wedge^n(T_{\gamma_y}\Gamma_X), \qquad \gamma_y := \gamma - \varepsilon_x + \varepsilon_y.$$

This is a section of the Hilbert bundle:

$$\wedge^n(T_{\gamma_y}\Gamma_X) \mapsto y \in \mathcal{O}_{\gamma,x}. \tag{2.6}$$

The Levi–Civita connection on TX generates in a natural way a connection on this bundle. We denote by $\nabla^X_{\gamma,x}$ the corresponding covariant derivative and use the notation

$$\nabla^X_x W(\gamma) \nabla^X_{\gamma,x} W_x(\gamma, x) \in T_x X \otimes (\wedge^n(T_\gamma \Gamma_X))$$

if the section $W_x(\gamma, \cdot)$ is differentiable at x.

We say that the form W is differentiable at a point γ if for each $x \in \gamma$ the section $W_x(\gamma, \cdot)$ is differentiable at x, and

$$\nabla^\Gamma W(\gamma) := (\nabla^X_x W(\gamma))_{x\in\gamma} \in T_\gamma\Gamma_X \otimes (\wedge^n(T_\gamma \Gamma_X)).$$

The mapping

$$\Gamma_X \ni \gamma \mapsto \nabla^\Gamma W(\gamma) \in T_\gamma\Gamma_X \otimes (\wedge^n(T_\gamma \Gamma_X))$$

will be called the covariant gradient of the form W.

Analogously, one can introduce higher order derivatives of a differential form W. Precisely, the kth derivative $(\nabla^\Gamma)^{(k)} W(\gamma)$ belongs to $(T_\gamma\Gamma_X)^{\otimes k} \otimes (\wedge^n(T_\gamma \Gamma_X))$.

Let us note that, for any $\eta \subset \gamma$, the space $\wedge^n(T_\eta \Gamma_X)$ can be identified in a natural way with a subspace of $\wedge^n(T_\gamma \Gamma_X)$. In this sense, we will use the expression $W(\gamma) = W(\eta)$ without additional explanations.

A form $W : \Gamma_X \to \wedge^n(T\Gamma_X)$ is called local if there exists $\Lambda = \Lambda(W) \in \mathcal{O}_c(X)$ such that $W(\gamma) = W(\gamma_\Lambda)$ for each $\gamma \in \Gamma_X$.

Let $\mathcal{F}\Omega^n$ denote the set of all local, infinitely differentiable forms $W : \Gamma_X \to \wedge^n(T\Gamma_X)$ such that there exist $\varphi \in C_0(X)$, $\varphi \geq 0$, and $l \in \mathbb{N}$ (depending on W) satisfying:

$$\|W(\gamma)\|^2_{\wedge^n(T_\gamma \Gamma_X)} \leq \langle \varphi, \gamma \rangle^l \quad \text{for all } \gamma \in \Gamma_X. \tag{2.7}$$

Below, we will give an explicit construction of a class of forms belonging to $\mathcal{F}\Omega^n$.

Let μ be a probability measure on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ which has all moments finite, i.e.:

$$\forall k \in \mathbb{N}, \forall \varphi \in C_0(X), \varphi \geq 0 : \qquad \int_{\Gamma_X} \langle \varphi, \gamma \rangle^k \mu(\mathrm{d}\gamma) < \infty. \tag{2.8}$$

Our next goal is to give a description of the space of n-forms that are square integrable with respect to the measure μ.

Let $\widetilde{\mathcal{F}\Omega^n}^\mu$ denote the μ-classes determined by $\mathcal{F}\Omega^n$. We define on $\widetilde{\mathcal{F}\Omega^n}^\mu$ the L^2-scalar product with respect to the measure μ:

$$(W_1, W_2)_{L^2_\mu \Omega^n} := \int_{\Gamma_X} \langle W_1(\gamma), W_2(\gamma) \rangle_{\wedge^n(T_\gamma \Gamma_X)} \, \mu(\mathrm{d}\gamma). \tag{2.9}$$

The integral on the right hand side of (2.9) is finite because of (2.7) and (2.8). Now, we define the Hilbert space $L^2_\mu \Omega^n = L^2(\Gamma_X \to \wedge^n(T\Gamma_X); \mu)$ as the completion of $\widetilde{\mathcal{F}\Omega^n}^\mu$ with respect to the norm generated by the scalar product (2.9). In what follows, we will not distinguish in notations between $\mathcal{F}\Omega^n$ and $\widetilde{\mathcal{F}\Omega^n}^\mu$, since it will be clear from the context which of these sets we mean.

Let m denote the volume measure on X. From now on, we suppose that, for any measurable function $F : \Gamma_X \times X \to \mathbb{R}$, $F \geq 0$:

$$\int_{\Gamma_X} \mu(\mathrm{d}\gamma) \int_X \gamma(\mathrm{d}x) F(\gamma, x) = \int_{\Gamma_X} \mu(\mathrm{d}\gamma) \int_X \sigma(\gamma, \mathrm{d}x) F(\gamma + \varepsilon_x, x), \tag{2.10}$$

where $\sigma(\gamma, \cdot) \ll m$ for μ-a.e. $\gamma \in \Gamma_X$. We shall use the notation

$$\rho(\gamma, x) := \frac{\mathrm{d}\sigma(\gamma, \cdot)}{\mathrm{d}m}(x).$$

In the theory of point processes, this property of the measure μ is called Σ'_m (see [42]). All Gibbs measures, in particular, all Poisson measures satisfy this property (see [26,43,45]). We consider this case in Section 4.

We will need the following consequence of the property Σ'_m. Let $: \gamma^{\otimes k} :$ be the measure on X^k given by

$$: \gamma^{\otimes k} : (\mathrm{d}x_1, \dots, \mathrm{d}x_k) := \sum_{\{y_1, \dots, y_k\} \subset \gamma} \varepsilon_{y_1} \hat{\otimes} \cdots \hat{\otimes} \varepsilon_{y_k} (\mathrm{d}x_1, \dots, \mathrm{d}x_k),$$

where

$$\varepsilon_{y_1}\hat{\otimes}\cdots\hat{\otimes}\varepsilon_{y_k}(\mathrm{d}x_1,\dots,\mathrm{d}x_k) := \frac{1}{k!}\sum_{\sigma\in S_k}\varepsilon_{y_{\sigma(1)}}\otimes\cdots\otimes\varepsilon_{y_{\sigma(k)}}(\mathrm{d}x_1,\dots,\mathrm{d}x_k),$$

S_k denoting the group of all permutations of $\{1,\dots,k\}$.

For μ-a.e. $\gamma\in\Gamma_X$, we denote by $\sigma^{(k)}(\gamma,\cdot)$ the measure on X^k given by

$$\sigma^{(k)}(\gamma,\mathrm{d}x_1,\dots,\mathrm{d}x_k) := \sigma(\gamma,\mathrm{d}x_1)\sigma(\gamma+\varepsilon_{x_1},\mathrm{d}x_2)\cdots\sigma(\gamma+\varepsilon_{x_1}+\cdots+\varepsilon_{x_{k-1}},\mathrm{d}x_k)$$

and let $\mu^{(k)}$ be the measure on $\Gamma_X\times X^k$ defined by

$$\mu^{(k)}(\mathrm{d}\gamma,\mathrm{d}x_1,\dots,\mathrm{d}x_k) := \mu(\mathrm{d}\gamma)\sigma^{(k)}(\gamma,\mathrm{d}x_1,\dots,\mathrm{d}x_k).$$

Lemma 2.1. *For any measurable $F:\Gamma_X\times X^k\to\mathbb{R}$, $F\geq 0$, $k\in\mathbb{N}$:*

$$\begin{aligned}&k!\int_{\Gamma_X}\mu(\mathrm{d}\gamma)\int_{X^k}:\gamma^{\otimes k}:(\mathrm{d}x_1,\dots,\mathrm{d}x_k)F(\gamma,x_1,\dots,x_k)\\&\quad=\int_{\Gamma_X\times X^k}\mu^{(k)}(\mathrm{d}\gamma,\mathrm{d}x_1,\dots,\mathrm{d}x_k)F(\gamma+\varepsilon_{x_1}+\cdots+\varepsilon_{x_k},x_1,\dots,x_k).\end{aligned}\tag{2.11}$$

Proof. We prove this by induction. For $k=1$, (2.11) is just (2.10). Let us suppose that (2.11) holds up to $k-1$. As easily seen:

$$k:\gamma^{\otimes k}:(\mathrm{d}x_1,\dots,\mathrm{d}x_k)=\gamma(\mathrm{d}x_k):(\gamma-\varepsilon_{x_k})^{\otimes k-1}:(\mathrm{d}x_1,\dots,\mathrm{d}x_{k-1}).$$

Then, by the induction hypothesis we have

$$\begin{aligned}&k!\int_{\Gamma_X}\mu(\mathrm{d}\gamma)\int_{X^k}:\gamma^{\otimes k}:(\mathrm{d}x_1,\dots,\mathrm{d}x_k)F(\gamma,x_1,\dots,x_k)\\&\quad=\int_{\Gamma_X}\mu(\mathrm{d}\gamma)\int_X\gamma(\mathrm{d}x_k)(k-1)!\int_{X^{k-1}}:(\gamma-\varepsilon_{x_k})^{\otimes(k-1)}:(\mathrm{d}x_1,\dots,\mathrm{d}x_{k-1})\\&\qquad\times F(\gamma,x_1,\dots,x_k)\\&\quad=\int_{\Gamma_X}\mu(\mathrm{d}\gamma)\int_X\sigma(\gamma,\mathrm{d}x_k)\int_{X^{k-1}}:\gamma^{\otimes(k-1)}:(\mathrm{d}x_1,\dots,\mathrm{d}x_{k-1})F(\gamma+\varepsilon_{x_k},x_1,\dots,x_k)\\&\quad=\int_{\Gamma_X}\mu(\mathrm{d}\gamma)\int_{X^{k-1}}:\gamma^{\otimes k-1}:(\mathrm{d}x_1,\dots,\mathrm{d}x_{k-1})\int_X\sigma(\gamma,\mathrm{d}x_k)F(\gamma+\varepsilon_{x_k},x_1,\dots,x_k)\\&\quad=\int_{\Gamma_X}\mu(\mathrm{d}\gamma)\int_X\sigma(\gamma,\mathrm{d}x_1)\int_X\sigma(\gamma+\varepsilon_{x_1},\mathrm{d}x_2)\cdots\int_X\sigma(\gamma+\varepsilon_{x_1}+\cdots+\varepsilon_{x_{k-1}},\mathrm{d}x_k)\\&\qquad\times F(\gamma+\varepsilon_{x_1}+\cdots+\varepsilon_{x_k},x_1,\dots,x_k).\end{aligned}$$

□

We will now give an isomorphic description of the space $L^2_\mu\Omega^n$. We first need some preparations. Let

$$\tilde{X}^k := \{(x_1,\dots,x_k)\in X^k : x_i\neq x_j \text{ if } i\neq j\}.$$

Notice that the set $X^k \setminus \tilde{X}^k$ is of zero $m^{\otimes k}$ measure. We have, for each $(x_1, \dots, x_k) \in \tilde{X}^k$:

$$\wedge^n (T_{(x_1,\dots,x_k)} X^k) = \wedge^n \left(\oplus_{i=1}^k T_{x_i} X \right) = \bigoplus_{\substack{0 \le l_1,\dots,l_k \le d \\ l_1+\cdots+l_k=n}} (T_{x_1} X)^{\wedge l_1} \wedge \cdots \wedge (T_{x_k} X)^{\wedge l_k}. \tag{2.12}$$

For a form $\omega : X^k \to \wedge^n (TX^k)$ and $(x_1, \dots, x_k) \in \tilde{X}^k$, we denote by $\omega(x_1, \dots, x_k)_{l_1,\dots,l_k}$ the corresponding component of $\omega(x_1, \dots, x_k)$ in the decomposition (2.12).

We introduce a set $\Psi^n_{\mathrm{sym}}(X^k)$ of smooth forms $\omega : X^k \to \wedge^n (TX^k)$ which have compact support and satisfy on $\tilde{X}^k$ the following assumptions:

(i) $\omega(x_1, \dots, x_k)_{l_1,\dots,l_k} = 0$ if $l_j = 0$ for some $j \in \{1, \dots, k\}$.

(ii) ω is invariant under the action of the group S_k:

$$\omega(x_1, \dots, x_k) = \omega(x_{\sigma(1)}, \dots, x_{\sigma(k)}) \quad \text{for each } \sigma \in S_k \tag{2.13}$$

(we identify the spaces $T_{(x_1,\dots,x_k)} X^k = \oplus_{i=1}^k T_{x_i} X$ and $T_{(x_{\sigma(1)},\dots,x_{\sigma(k)})} X^k = \oplus_{i=1}^k T_{x_{\sigma(i)}}$ through the natural isomorphism).

Using (2.8) and Lemma 2.1, we easily conclude that any mapping of the form:

$$\Gamma_X \times X^k \ni (\gamma, x_1, \dots, x_k) \mapsto F(\gamma)\omega(x_1, \dots, x_k) \in \wedge^n (T_{(x_1,\dots,x_k)} X^k), \tag{2.14}$$

where $F \in \mathcal{F}C^\infty_{\mathrm{b}}(\mathcal{D}, \Gamma_X)$ and $\omega \in \Psi^n_{\mathrm{sym}}(X^k)$ belongs to the space $L^2(\Gamma_X \times X^k \to \wedge^n (TX^k); \mu^{(k)})$. Let $L^2_\Psi(\Gamma_X \times X^k \to \wedge^n (TX^k); \mu^{(k)})$ denote the closed linear span of all mappings of the form (2.14) in $L^2(\Gamma_X \times X^k \to \wedge^n (TX^k); \mu^{(k)})$. It is not hard to show that the latter is just the space of all $\mu^{(k)}$-square-integrable mappings of the form:

$$\Gamma_X \times \tilde{X}^k \ni (\gamma, x_1, \dots, x_k) \mapsto \mathcal{W}(\gamma, x_1, \dots, x_k) \in \mathbb{T}^{(n)}_{\{x_1,\dots,x_k\}} X^k$$

such that, for $\mu^{(k)}$-a.e. $(\gamma, x_1, \dots, x_k) \in \Gamma_X \times \tilde{X}^k$:

$$\mathcal{W}(\gamma, x_1, \dots, x_k) = \mathcal{W}(\gamma, x_{\sigma(1)}, \dots, x_{\sigma(k)}), \quad \sigma \in S_k.$$

Here:

$$\mathbb{T}^{(n)}_{\{x_1,\dots,x_k\}} X^k := \bigoplus_{\substack{1 \le l_1,\dots,l_k \le d \\ l_1+\cdots+l_k=n}} (T_{x_1} X)^{\wedge l_1} \wedge \cdots \wedge (T_{x_k} X)^{\wedge l_k} \tag{2.15}$$

(Notice that the space $\mathbb{T}^{(n)}_{\{x_1,\dots,x_k\}} X^k$ is indeed independent of the order of the points $x_1, \dots, x_k$.)

Remark 2.2. Evidently:

$$L^2_\Psi(\Gamma_X \times X^k \mapsto \wedge^n (TX^k); \mu^{(k)}) = L^2_\mu \left(\Gamma_X \to \bigcup_{\gamma \in \Gamma_X} L^2_{\sigma^{(m)}(\gamma)} \Psi^n_{\mathrm{sym}}(X^m) \right),$$

where the latter space was defined in Section 1 (see formulas (1.2) and (1.3)).

By virtue of (2.4) and (2.15), we have

$$\wedge^n(T_\gamma \Gamma_X) = \oplus_{k=1}^n \oplus_{\{x_1,\dots,x_k\}\subset\gamma} \mathbb{T}^{(n)}_{\{x_1,\dots,x_k\}} X^k. \tag{2.16}$$

For $W \in \Gamma_X \to \wedge^n(T\Gamma_X)$, we denote by $W_k(\gamma) \in \oplus_{\{x_1,\dots,x_k\}\subset\gamma}\mathbb{T}^{(n)}_{\{x_1,\dots,x_k\}}X^k$ the corresponding component of $W(\gamma) \in \wedge^n(T_\gamma\Gamma_X)$ in the decomposition (2.16). Thus, for $\{x_1,\dots,x_k\} \subset \gamma$, $W_k(\gamma, x_1,\dots,x_k)$ is equal to the projection of $W(\gamma)$ onto the subspace $\mathbb{T}^{(n)}_{\{x_1,\dots,x_k\}}X^k$.

Proposition 2.3. *The space $L^2_\mu\Omega^n$ is unitarily isomorphic to the space*

$$\bigoplus_{k=1}^n L^2_\Psi(\Gamma_X \times X^k \to \wedge^n(TX^k); \mu^{(k)}), \tag{2.17}$$

where the corresponding isomorphism I^n is defined by the formula

$$I^n_k W(\gamma, x_1,\dots,x_k) := (k!)^{-1/2} W_k(\gamma+\varepsilon_{x_1}+\dots+\varepsilon_{x_k}, x_1,\dots,x_k), \quad k=1,\dots,n. \tag{2.18}$$

Here, $I^n_k W := (I^n W)_k$ is the k-th component of $I^n W$ in the decomposition (2.17).

Proof. A direct calculation shows that

$$\|W(\gamma)\|^2_{\wedge^n(T_\gamma\Gamma_X)} = \sum_{k=1}^n \int_{X^k} \|W_k(\gamma, x_1,\dots,x_k)\|^2_{\mathbb{T}^{(n)}_{\{x_1,\dots,x_k\}}X^k} : \gamma^{\otimes k} : (\mathrm{d}x_1,\dots,\mathrm{d}x_k). \tag{2.19}$$

Therefore, by Lemma 2.1, we have for any $W \in \mathcal{F}\Omega^n$:

$$\begin{aligned}\int_{\Gamma_X} &\|W(\gamma)\|^2_{\wedge^n(T_\gamma\Gamma_X)}\mu(\mathrm{d}\gamma)\\ &= \sum_{k=1}^n \int_{\Gamma_X\times X^k} \|W_k(\gamma+\varepsilon_{x_1}+\dots+\varepsilon_{x_k}, x_1,\dots,x_k)\|^2_{\mathbb{T}^{(n)}_{\{x_1,\dots,x_k\}}X^k}\\ &\quad\times \mu^{(k)}(\mathrm{d}\gamma, \mathrm{d}x_1,\dots,\mathrm{d}x_k).\end{aligned}$$

Hence, I^n is an isometry of the space $L^2_\mu\Omega^n$ into the space (2.17). Next, the image of each mapping (2.14) under $(I^{(n)})^{-1}$ is given by

$$W_l(\gamma, x_1,\dots,x_l) := \begin{cases} 0, & l \neq k,\\ (k!)^{1/2} F(\gamma - \varepsilon_{x_1} - \dots - \varepsilon_{x_k})\omega(x_1,\dots,x_k), & l = k \end{cases} \tag{2.20}$$

and evidently belongs to $\mathcal{F}\Omega^n$. Therefore, I^n is "onto". $\square$

In what follows, we will denote by $\mathcal{D}\Omega^n$ the linear span of the forms defined by (2.20) with $k = 1,\dots,n$. As we already noticed in the proof of Proposition 2.3, $\mathcal{D}\Omega^n$ is a subset of $\mathcal{F}\Omega^n$ and is dense in $L^2_\mu\Omega^n$.

3. Laplace operators on differential forms over configuration spaces

In this section, we introduce differential operators associated with the measure μ on Γ_X which act in the space of square-integrable forms. These operators generalize the notions of Bochner and deRham Laplacians on finite-dimensional manifolds. But first, we consider the Dirichlet operator in the space $L^2(\Gamma_X;\mu)$.

3.1. Dirichlet operator on functions

For each $\gamma \in \Gamma_X$, consider the triple:

$$T_{\gamma,\infty}\Gamma_X \supset T_\gamma \Gamma_X \supset T_{\gamma,0}\Gamma_X.$$

Here, $T_{\gamma,0}\Gamma_X$ consists of all finite sequences from $T_\gamma\Gamma_X$, and $T_{\gamma,\infty}\Gamma_X := (T_{\gamma,0}\Gamma_X)'$ is the dual space, which consists of all sequences $V(\gamma) = (V(\gamma,x))_{x\in\gamma}$, where $V(\gamma,x) \in T_xX$. The pairing between any $V(\gamma) \in T_{\gamma,\infty}\Gamma_X$ and $v(\gamma) \in T_{\gamma,0}\Gamma_X$ with respect to the zero space $T_\gamma\Gamma_X$ is given by

$$\langle V(\gamma), v(\gamma)\rangle_\gamma = \sum_{x\in\gamma} \langle V(\gamma,x), v(\gamma,x)\rangle_x$$

(the series is, in fact, finite). From now on, under a vector field over Γ_X we will understand mappings of the form $\Gamma_X \ni \gamma \mapsto V(\gamma) \in T_{\gamma,\infty}\Gamma_X$.

We will suppose that, for $\mu\otimes m$-a.e. $(\gamma,x) \in \Gamma_X \times X$, $\rho(\gamma,x) > 0$ and for μ-a.e. $\gamma \in \Gamma_X$, the function $\rho(\gamma,\cdot)$ is weakly differentiable on X. We set

$$\beta_\sigma(\gamma,x) := \frac{\nabla_x^X \rho(\gamma,x)}{\rho(\gamma,x)}, \quad \mu\otimes m\text{-a.e. } (\gamma,x)\in\Gamma_X\times X$$

($\beta_\sigma(\gamma,\cdot)$ is called the logarithmic derivative of the measure $\sigma(\gamma,\cdot)$).

The logarithmic derivative of the measure μ is set to be the μ-a.e. defined vector field on Γ_X given by

$$\gamma \mapsto B_\mu(\gamma) = (B_\mu(\gamma,x))_{x\in\gamma} \in T_{\gamma,\infty}\Gamma_X, \quad B_\mu(\gamma,x) := \beta_\sigma(\gamma-\varepsilon_x, x).$$

We define a bilinear form $\mathcal{E}_\mu$ on the space $L^2(\Gamma_X;\mu)$ by setting

$$\mathcal{E}_\mu(F^{(1)},F^{(2)}) := \int_{\Gamma_X} \langle \nabla^\Gamma F^{(1)}(\gamma), \nabla^\Gamma F^{(2)}(\gamma)\rangle_\gamma \mu(\mathrm{d}\gamma), \tag{3.1}$$

where $F^{(1)}, F^{(2)} \in D(\mathcal{E}_\mu) := \mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma_X)$. By (2.3) and (2.8), and [40, Theorem 2.4], $(\mathcal{E}_\mu, \mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma_X))$ is a pre-Dirichlet form.

Theorem 3.1. *Suppose that, for any $\Lambda \in \mathcal{O}_{\mathrm{c}}(X)$:*

$$\int_{\Gamma_X} \left(\sum_{x\in\gamma_\Lambda} |B_\mu(\gamma,x)|_x\right)^2 \mu(\mathrm{d}\gamma) < \infty. \tag{3.2}$$

Then, for any $F^{(1)}, F^{(2)} \in \mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D}, \Gamma_X)$*, we have*

$$\mathcal{E}_\mu(F^{(1)}, F^{(2)}) = \int_{\Gamma_X} (\mathbf{H}_\mu F^{(1)})(\gamma) F^{(2)}(\gamma)\mu(\mathrm{d}\gamma), \tag{3.3}$$

where $\mathbf{H}_\mu$ *is the operator in the space* $L^2(\Gamma_X; \mu)$ *with domain* $\mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D}, \Gamma_X)$ *given by*

$$(\mathbf{H}_\mu F)(\gamma) := -\Delta^\Gamma F(\gamma) - \langle \nabla^\Gamma F(\gamma), B_\mu(\gamma)\rangle_\gamma, \quad F \in \mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D}, \Gamma_X). \tag{3.4}$$

Here:

$$\Delta^\Gamma F(\gamma) := \sum_{x\in\gamma} \Delta_x^X F(\gamma), \quad \Delta_x^X F(\gamma) := \Delta^X F_x(\gamma, x), \tag{3.5}$$

where Δ^X *denotes the Laplacian on* X *corresponding to the volume measure* m.

Corollary 3.2. $(\mathcal{E}_\mu, \mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D}, \Gamma_X))$ *is closable on* $L^2(\Gamma_X; \mu)$*. Its closure, denoted by* $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$*, is associated with a positive definite self-adjoint operator, the Friedrichs extension of* $\mathbf{H}_\mu$*, which we also denote by* $\mathbf{H}_\mu$.

Remark 3.3. In case of a Ruelle measure, a theorem on the L^2-generator of the bilinear form (3.1) was proved in [10]. A theorem on the closability of the form (3.1) in the case of a Gibbs measure on a manifold X was proved in [20] and in the general case of a Σ'_m-measure in [40] (see also [39]).

Proof of Theorem 3.1. First, we note that, for each $F \in \mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D}, \Gamma_X)$ and each $\gamma \in \Gamma_X$, the function $f(x) := F(\gamma + \varepsilon_x) - F(\gamma)$ belongs to $\mathcal{D}$ and $\nabla^X f(x) = \nabla_x^X F(\gamma + \varepsilon_x)$.

Let now $F^{(1)}, F^{(2)} \in \mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D}, \Gamma_X)$ and let $\Lambda \in \mathcal{O}_{\mathrm{c}}(X)$ be such that there exits a compact $\Lambda' \subset \Lambda$ satisfying $F^{(i)}(\gamma) = F^{(i)}(\gamma_{\Lambda'})$, $i = 1, 2$, for all $\gamma \in \Gamma_X$. Then, by (2.10)

$$\begin{aligned}
&\int_{\Gamma_X} \langle \nabla^\Gamma F^{(1)}(\gamma), \nabla^\Gamma F^{(2)}(\gamma)\rangle_\gamma \mu(\mathrm{d}\gamma)\\
&\quad = \int_{\Gamma_X} \mu(\mathrm{d}\gamma) \int_\Lambda m(\mathrm{d}x)\rho(\gamma, x)\langle \nabla_x^X F^{(1)}(\gamma + \varepsilon_x), \nabla_x^X F^{(2)}(\gamma + \varepsilon_x)\rangle_x\\
&\quad = -\int_{\Gamma_X} \mu(\mathrm{d}\gamma) \int_\Lambda m(\mathrm{d}x)\rho(\gamma, x)(\Delta_x^X F^{(1)}(\gamma + \varepsilon_x) + \langle \nabla_x^X F^{(1)}(\gamma + \varepsilon_x), \beta_\sigma(\gamma, x)\rangle_x)\\
&\qquad \times F^{(2)}(\gamma + \varepsilon_x)\\
&\quad = -\int_\Gamma \mu(\mathrm{d}\gamma) \sum_{x\in\gamma_\Lambda} (\Delta_x^X F^{(1)}(\gamma) + \langle \nabla_x^X F^{(1)}(\gamma), B_\mu(\gamma, x)\rangle_x) F^{(2)}(\gamma)\\
&\quad = \int_\Gamma (\mathbf{H}_\mu F^{(1)})(\gamma) F^{(2)}(\gamma).
\end{aligned}$$

As easily seen, condition (3.2) guarantees the inclusion $\mathbf{H}_\mu F^{(1)} \in L^2(\Gamma; \mu)$. □

3.2. Bochner Laplacian on forms

Let us consider the bilinear form $\mathcal{E}^{\mathrm{B}}_{\mu,n}$ defined by

$$\mathcal{E}^{\mathrm{B}}_{\mu,n}(W^{(1)}, W^{(2)}) = \int_{\Gamma_X} \langle \nabla^\Gamma W^{(1)}(\gamma), \nabla^\Gamma W^{(2)}(\gamma)\rangle_{T_\gamma\Gamma_X \otimes \wedge^n(T_\gamma\Gamma_X)}\mu(\mathrm{d}\gamma), \tag{3.6}$$

where $W^{(1)}, W^{(2)} \in D(\mathcal{E}^{\mathrm{B}}_{\mu,n}) := \mathcal{D}\Omega^n$. It follows from the definition of $\mathcal{D}\Omega^n$ that, for each $W \in \mathcal{D}\Omega^n$, there exists $\varphi \in \mathcal{D}$, $\varphi \geq 0$, such that

$$\|\nabla^\Gamma W(\gamma)\|^2_{T_\gamma\Gamma_X \otimes \wedge^n(T_\gamma\Gamma_X)} \leq \langle \varphi, \gamma\rangle^{n+1} \quad \text{for all } \gamma \in \Gamma_X \tag{3.7}$$

and therefore, by (2.8), the function under the sign of integral in (3.6) is integrable with respect to μ.

The following lemma shows that the bilinear form $(\mathcal{E}^{\mathrm{B}}_{\mu,n}, \mathcal{D}\Omega^n)$ is well defined on $L^2\Omega^n$.

Lemma 3.4. *We have $\mathcal{E}^{\mathrm{B}}_{\mu,n}(W^{(1)}, W^{(2)}) = 0$ for all $W^{(1)}, W^{(2)} \in \mathcal{D}\Omega^n$ such that $W^{(1)} = 0$ μ-a.e.*

Proof. Let $W \in \mathcal{D}\Omega^n$ and $W = 0$ μ-a.e. For $x_0 \in X$ and $R > 0$, let

$$B(x_0, R) := \{x \in X | d(x_0, x) < R\},$$

where $d(\cdot, \cdot)$ denotes the Riemannian distance on X. Then

$$\begin{aligned}0 &= \int_{\Gamma_X} \mu(\mathrm{d}\gamma) \int_{B(x_0,R)} \gamma(\mathrm{d}x)\|W(\gamma)\|_{\wedge^n(T_\gamma\Gamma_X)}\\ &= \int_{\Gamma_X} \mu(\mathrm{d}\gamma) \int_{B(x_0,R)} m(\mathrm{d}x)\rho(\gamma, x)\|W(\gamma + \varepsilon_x)\|_{\wedge^n(T_{\gamma+\varepsilon_x}\Gamma_X)}.\end{aligned}$$

Since R was arbitrary, we therefore have

$$\|W(\gamma + \varepsilon_x)\|_{\wedge^n(T_{\gamma+\varepsilon_x}\Gamma_X)} = 0, \quad \mu \otimes m\text{-a.e. } (\gamma, x) \in \Gamma_X \times X.$$

For a fixed $\gamma \in \Gamma_X$, the function $X \setminus \gamma \ni x \mapsto \|W(\gamma + \varepsilon_x)\|_{\wedge^n(T_{\gamma+\varepsilon_x}\Gamma_X)}$ is continuous, and therefore for μ-a.e. $\gamma \in \Gamma_X$, $W(\gamma + \varepsilon_x) = 0$ on $X \setminus \gamma$. Hence

$$\begin{aligned}\mathcal{E}^{\mathrm{B}}_{\mu,n}(W, W) &= \int_{\Gamma_X} \mu(\mathrm{d}\gamma) \int_X \gamma(\mathrm{d}x)\|\nabla^X_x W(\gamma)\|^2_{T_x X \otimes \wedge^n(T_\gamma\Gamma_X)}\\ &= \int_{\Gamma_X} \mu(\mathrm{d}\gamma) \int_X m(\mathrm{d}x)\rho(\gamma, x)\|\nabla^X_x W(\gamma + \varepsilon_x)\|^2_{T_x(X)\otimes\wedge^n(T_{\gamma+\varepsilon_x}\Gamma_X)} = 0.\end{aligned}$$

From here the lemma follows by the Schwarz inequality. □

Theorem 3.5. *Suppose that*

$$\forall \Lambda \in \mathcal{O}_{\mathrm{c}}(X)\, \exists \varepsilon > 0 : \int_{\Gamma_X} \left(\sum_{x\in\gamma_\Lambda} |B_\mu(\gamma, x)|_x\right)^{2+\varepsilon} \mu(\mathrm{d}\gamma) < \infty. \tag{3.8}$$

Then, for any $W^{(1)}, W^{(2)} \in \mathcal{D}\Omega^n$, we have

$$\mathcal{E}^{\mathrm{B}}_{\mu,n}(W^{(1)}, W^{(2)}) = \int_{\Gamma_X} \langle \mathbf{H}^{\mathrm{B}}_{\mu,n} W^{(1)}(\gamma), W^{(2)}(\gamma)\rangle_{\wedge^n(T_\gamma\Gamma_X)}\mu(\mathrm{d}\gamma),$$

where $\mathbf{H}^{\mathrm{B}}_{\mu,n}$ is the operator in the space $L^2_\mu\Omega^n$ with domain $\mathcal{D}\Omega^n$ given by

$$\mathbf{H}^{\mathrm{B}}_{\mu,n} W(\gamma) := -\Delta^\Gamma W(\gamma) - \langle \nabla^\Gamma W(\gamma), B_\mu(\gamma)\rangle_\gamma, \quad W \in \mathcal{D}\Omega^n. \tag{3.9}$$

Here:

$$\Delta^\Gamma W(\gamma) := \sum_{x\in\gamma} \Delta^X_x W(\gamma), \tag{3.10}$$

where Δ^X_x is the Bochner Laplacian of the bundle $\wedge^n(T_{\gamma_y}\Gamma_X) \mapsto y \in \mathcal{O}_{\gamma,x}$ with the volume measure m.

Proof. We first note that, for any $W \in \mathcal{D}\Omega^n$, the form $\mathbf{H}^{\mathrm{B}}_{\mu,n} W$ defined by (3.9) and (3.10) belongs to $L^2_\mu\Omega^n$. Indeed, as easily seen, $\Delta^\Gamma W \in \mathcal{F}\Omega^n$, and hence $\Delta^\Gamma W \in L^2_\mu\Omega^n$. Next, choose any $\Lambda \in \mathcal{O}_{\mathrm{c}}(X)$ such that there exists a compact $\Lambda' \subset \Lambda$ satisfying $W(\gamma) = W(\gamma_{\Lambda'})$ for all $\gamma \in \Gamma_X$. Then:

$$\begin{aligned}
&\int_{\Gamma_X} \|\langle \nabla^\Gamma W(\gamma), B_\mu(\gamma)\rangle_\gamma\|^2_{\wedge^n(T_\gamma\Gamma_X)}\mu(\mathrm{d}\gamma) \\
&\quad = \int_{\Gamma_X} \left\| \sum_{x\in\gamma_\Lambda} \langle \nabla^X_x W(\gamma), B_\mu(\gamma, x)\rangle_x \right\|^2_{\wedge^n(T_\gamma\Gamma_X)} \mu(\mathrm{d}\gamma) \\
&\quad \le \int_{\Gamma_X} \left(\sum_{x\in\gamma_\Lambda} \|\nabla^X_x W(\gamma)\|_{T_xX\otimes\wedge^n(T_\gamma\Gamma_X)} |B_\mu(\gamma, x)|_x \right)^2 \mu(\mathrm{d}\gamma).
\end{aligned} \tag{3.11}$$

As easily seen, there exists $\varphi \in C_0(X)$, $\varphi \ge 0$, such that

$$\|\nabla^X_x W(\gamma)\|^2_{T_xX\otimes\wedge^n(T_\gamma\Gamma_X)} \le \langle\varphi, \gamma\rangle^n \quad \text{for all } \gamma \in \Gamma_X,\ x \in \gamma. \tag{3.12}$$

Now, by using (2.8), (3.8), (3.11) and (3.12), and the Schwarz inequality, we conclude that

$$\int_{\Gamma_X} \|\langle \nabla^\Gamma W(\gamma), B_\mu(\gamma)\rangle_\gamma\|^2_{\wedge^n(T_\gamma\Gamma_X)}\, \mu(\mathrm{d}\gamma) < \infty. \tag{3.13}$$

Next, we will need the following lemma, whose proof follows directly from the construction of the forms from $\mathcal{D}\Omega^n$.

Lemma 3.6. *For each fixed $W \in \mathcal{D}\Omega^n$ and $\gamma \in \Gamma_X$, the mapping*

$$X \setminus \gamma \ni x \mapsto \omega(x) := W(\gamma + \varepsilon_x) \in \wedge^n(T_{\gamma+\varepsilon_x}\Gamma_X) = \wedge^n(T_\gamma\Gamma_X \oplus T_xX)$$

(*uniquely*) *extends to a smooth form*

$$X \ni x \mapsto \omega(x) \in \wedge^n(T_\gamma\Gamma_X \oplus T_xX),$$

and $\nabla^X\omega = 0$ on $\Lambda^c := X\setminus\Lambda$, *where* $\Lambda\subset X$ *is compact and such that* $W(\gamma') = W(\gamma'_\Lambda)$ *for all* $\gamma'\in\Gamma_X$.

Let $W^{(1)}, W^{(2)}\in\mathcal{D}\Omega^n$ and let $\Lambda\in\mathcal{O}_{\mathrm{c}}(X)$ be such that there exits a compact $\Lambda'\subset\Lambda$ satisfying $W^{(i)}(\gamma) = W^{(i)}(\gamma_{\Lambda'})$, $i = 1, 2$, for all $\gamma\in\Gamma_X$. Then, by virtue of (2.10) and (3.13), and Lemma 3.6 we get, analogously to the proof of Theorem 3.1:

$$
\begin{aligned}
&\mathcal{E}^{\mathrm{B}}_{\mu,n}(W^{(1)}, W^{(2)})\\
&\quad = \int_{\Gamma_X}\mu(\mathrm{d}\gamma)\int_\Lambda \gamma(\mathrm{d}x)\langle\nabla^X_x W^{(1)}(\gamma), \nabla^X_x W^{(2)}(\gamma)\rangle_{T_xX\otimes\wedge^n(T_\gamma\Gamma_X)}\\
&\quad = \int_{\Gamma_X}\mu(\mathrm{d}\gamma)\int_\Lambda m(\mathrm{d}x)\rho(\gamma,x)\langle\nabla^X_x W^{(1)}(\gamma+\varepsilon_x), \nabla^X_x W^{(2)}(\gamma+\varepsilon_x)\rangle_{T_xX\otimes\wedge^n(T_{\gamma+\varepsilon_x})}\\
&\quad = -\int_{\Gamma_X}\mu(\mathrm{d}\gamma)\int_\Lambda m(\mathrm{d}x)\rho(\gamma,x)[\langle\Delta^X_x W^{(1)}(\gamma+\varepsilon_x), W^{(2)}(\gamma+\varepsilon_x)\rangle_{\wedge^n(T_{\gamma+\varepsilon_x})}\\
&\qquad + \langle\langle\nabla^X_x W^{(1)}(\gamma+\varepsilon_x), \beta_\sigma(\gamma,x)\rangle_x, W^{(2)}(\gamma+\varepsilon_x)\rangle_{\wedge^n(T_{\gamma+\varepsilon_x}\Gamma_X)}]\\
&\quad = -\int_{\Gamma_X}\mu(\mathrm{d}\gamma)\int_\Lambda \gamma(\mathrm{d}x)[\langle\Delta^X_x W^{(1)}(\gamma), W^{(2)}(\gamma)\rangle_{\wedge^n(T_\gamma\Gamma_X)}\\
&\qquad + \langle\langle\nabla^X_x W^{(1)}(\gamma), B_\mu(\gamma,x)\rangle_x, W^{(2)}(\gamma)\rangle_{\wedge^n(T_\gamma\Gamma_X)}]\\
&\quad = \int_{\Gamma_X}\langle\mathbf{H}^{\mathrm{B}}_{\mu,n}W^{(1)}(\gamma), W^{(2)}(\gamma)\rangle_{\wedge^n(T_\gamma\Gamma_X)}\mu(\mathrm{d}\gamma).
\end{aligned}
$$

Corollary 3.7. $(\mathcal{E}^{\mathrm{B}}_{\mu,n}, \mathcal{D}\Omega^n)$ *is closable on* $L^2_\mu\Omega^n$. *Its closure* $(\mathcal{E}^{\mathrm{B}}_{\mu,n}, D(\mathcal{E}^{\mathrm{B}}_{\mu,n}))$ *is associated with a positive definite, self-adjoint operator, the Friedrichs extension of* $\mathbf{H}^{\mathrm{B}}_{\mu,n}$, *which we also denote by* $\mathbf{H}^{\mathrm{B}}_{\mu,n}$.

We define $(\tilde{\mathcal{E}}^{\mathrm{B}}_{\mu,n}, D(\tilde{\mathcal{E}}^{\mathrm{B}}_{\mu,n}))$ as the image of the bilinear form $(\mathcal{E}^{\mathrm{B}}_{\mu,n}, D(\mathcal{E}^{\mathrm{B}}_{\mu,n}))$ under the unitary I^n.

Proposition 3.8. *Let* $\mathcal{W}^{(1)}, \mathcal{W}^{(2)}\in I^n(\mathcal{D}\Omega^n)$. *Then*:

$$
\begin{aligned}
&\tilde{\mathcal{E}}^{\mathrm{B}}_{\mu,n}(\mathcal{W}^{(1)}, \mathcal{W}^{(2)})\\
&\quad = \sum_{k=1}^n\int_{\Gamma_X\times X^k}\mu^{(k)}(\mathrm{d}\gamma, \mathrm{d}x_1,\dots,\mathrm{d}x_k)\\
&\qquad \times[\langle\nabla^\Gamma_\gamma\mathcal{W}^{(1)}(\gamma,x_1,\dots,x_k), \nabla^\Gamma_\gamma\mathcal{W}^{(2)}(\gamma,x_1,\dots,x_k)\rangle_{T_\gamma\Gamma_X\otimes\mathbb{T}^{(n)}_{\{x_1,\dots,x_k\}}X^k}\\
&\qquad + \langle\nabla^{X^k}_{(x_1,\dots,x_k)}\mathcal{W}^{(1)}(\gamma,x_1,\dots,x_k),\\
&\qquad \nabla^{X^k}_{(x_1,\dots,x_k)}\mathcal{W}^{(2)}(\gamma,x_1,\dots,x_k)\rangle_{T_{(x_1,\dots,x_k)}X^k\otimes\mathbb{T}^{(n)}_{\{x_1,\dots,x_k\}}X^k}].
\end{aligned}
\tag{3.14}
$$

Here, for a fixed $(x_1,\dots,x_k)\in\tilde{X}^k$, ∇^Γ_γ *denotes the gradient of a mapping from* Γ *into* $\mathbb{T}^{(n)}_{\{x_1,\dots,x_k\}}X^k$ *defined for* μ*-a.e.* $\gamma\in\Gamma_X$ *similar to the gradient of a function on* Γ.

Proof. Let $W^{(1)}, W^{(2)} \in \mathcal{D}\Omega^n$ and let $\mathcal{W}^{(i)} := I^n W^{(i)}$, $i = 1, 2$. Then, by Lemma 2.1:

$$
\begin{aligned}
&\tilde{\mathcal{E}}^{\mathrm{B}}_{\mu,n}(\mathcal{W}^{(1)}, \mathcal{W}^{(2)}) \\
&\quad = \mathcal{E}^{\mathrm{B}}_{\mu,n}(W^{(1)}, W^{(2)}) = \int_{\Gamma_X} \mu(\mathrm{d}\gamma) \sum_{x\in\gamma} \langle \nabla^X_x W^{(1)}(\gamma), \nabla^X_x W^{(2)}(\gamma)\rangle_{T_xX\otimes\wedge^n(T_\gamma\Gamma_X)} \\
&\quad = \sum_{k=1}^{n} \int_{\Gamma_X} \mu(\mathrm{d}\gamma) \int_{X^k} : \gamma^{\otimes k} : (\mathrm{d}x_1, \dots, \mathrm{d}x_k) \\
&\qquad \times \sum_{x\in\gamma} \langle \nabla^X_x W^{(1)}_k(\gamma, x_1, \dots, x_k), \nabla^X_x W^{(2)}_k(\gamma, x_1, \dots, x_k)\rangle_{T_xX\otimes \mathbb{T}^{(n)}_{\{x_1,\dots,x_k\}}X^k} \\
&\quad = \sum_{k=1}^{n} \int_{\Gamma\times X^k} \mu^{(k)}(\mathrm{d}\gamma, \mathrm{d}x_1, \dots, \mathrm{d}x_k) \\
&\qquad \times \sum_{x\in\gamma\cup\{x_1,\dots,x_k\}} \langle \nabla^X_x \mathcal{W}^{(1)}(\gamma, x_1, \dots, x_k), \nabla^X_x \mathcal{W}^{(2)}(\gamma, x_1, \dots, x_k)\rangle_{T_xX\otimes \mathbb{T}^{(n)}_{\{x_1,\dots,x_k\}}X^k},
\end{aligned}
$$

which is equal to the right hand side of (3.14). □

We will now apply Proposition 3.8 to prove the vanishing of square-integrable Bochner-harmonic forms.

Theorem 3.9. *Let the conditions of Theorem 3.5 be satisfied, let*

$$\sigma^{(k)}(\gamma, X^k) = \infty \quad \text{for } \mu\text{-a.e. } \gamma \in \Gamma_X,\ k \in \mathbb{N} \tag{3.15}$$

and let one of the two following conditions hold:

(i) *For μ-a.e. $\gamma \in \Gamma_X$, $\rho(\gamma, \cdot)$ is continuous and positive on X.*
(ii) *$d \geq 2$ and for μ-a.e. $\gamma \in \Gamma_X$, $\rho(\gamma, \cdot)$ is continuous and positive on $X \setminus \gamma$.*

Then, for each $n \in \mathbb{N}$, $\operatorname{Ker} \mathbf{H}^{\mathrm{B}}_{\mu,n} = \{0\}$.

Proof. We will prove the theorem in the case of (ii), the case (i) being completely similar and simpler.

First, we note that we can suppose that, for all $\gamma \in \Gamma_X$, $\rho(\gamma, \cdot)$ is continuous and positive on $X \setminus \gamma$. It suffices to show that $\mathcal{E}^{\mathrm{B}}_{\mu,n}(W) = 0$, $W \in D(\mathcal{E}^{\mathrm{B}}_{\mu,n}) \Rightarrow W = 0$, or equivalently, $\tilde{\mathcal{E}}^{\mathrm{B}}_{\mu,n}(\mathcal{W}) = 0$, $\mathcal{W} \in D(\tilde{\mathcal{E}}^{\mathrm{B}}_{\mu,n}) \Rightarrow \mathcal{W} = 0$. Here and below, for a bilinear form E we set $E(W) := E(W, W)$ for $W \in D(E)$.

Let us consider the following bilinear form on the Hilbert space (2.17):

$$\mathcal{U}_n(\mathcal{W}^{(1)}, \mathcal{W}^{(2)}) := \sum_{k=1}^{n} \mathcal{U}_{k,n}(\mathcal{W}^{(1)}, \mathcal{W}^{(2)}),$$

$$\mathcal{U}_{k,n}(\mathcal{W}^{(1)},\mathcal{W}^{(2)}) := \int_{\Gamma_X\times X^k}\mu^{(k)}(\mathrm{d}\gamma,\mathrm{d}x_1,\dots,\mathrm{d}x_k)\langle\nabla^{X^k}_{(x_1,\dots,x_k)}\mathcal{W}^{(1)}(\gamma,x_1,\dots,x_k),\\ \nabla^{X^k}_{(x_1,\dots,x_k)}\mathcal{W}^{(2)}(\gamma,x_1,\dots,x_k)\rangle_{T_{(x_1,\dots,x_k)}X^k\otimes\mathbb{T}^{(n)}_{\{x_1,\dots,x_k\}}X^k},$$

$$\mathcal{W}^{(1)},\mathcal{W}^{(2)}\in I^n(\mathcal{D}\Omega^n).$$

From the existence of the generator of $\mathcal{U}_n$ defined on $I^n(\mathcal{D}\Omega^n)$, it follows that $\mathcal{U}_n$ is closable and let $(\mathcal{U}_n, D(\mathcal{U}_n))$ denote its closure. By Proposition 3.8, $D(\tilde{\mathcal{E}}^{\mathrm{B}}_{\mu,n})\subset D(\mathcal{U}_n)$ and $\tilde{\mathcal{E}}^{\mathrm{B}}_{\mu,n}(\mathcal{W})\geq\mathcal{U}_n(\mathcal{W})$ for all $\mathcal{W}\in D(\tilde{\mathcal{E}}^{\mathrm{B}}_{\mu,n})$. Furthermore, it follows from the definition of $\mathcal{U}_n$ that $D(\mathcal{U}_n)=\oplus_{k=1}^n D(\mathcal{U}_{k,n})$ and $\mathcal{U}_n=\sum_{k=1}^n\mathcal{U}_{k,n}$, where for each $k=1,\dots,n$ $(\mathcal{U}_{k,n}, D(\mathcal{U}_{k,n}))$ is a closed form on $L^2_\Psi(\Gamma_X\times X^k\to\wedge^n(TX^k);\mu^{(k)})=: H_{k,n}$. Hence, it suffices to show that $\mathcal{U}_{k,n}(\mathcal{W})=0$, $\mathcal{W}\in D(\mathcal{U}_{k,n})\Rightarrow\mathcal{W}=0$.

For $\mathcal{W}\in I^n(\mathcal{D}\Omega^n)\cap H_{k,n}=:\Omega_{k,n}$, we define

$$S(\mathcal{W})(\gamma,x_1,\dots,x_k) := \|\nabla^{X^k}_{(x_1,\dots,x_k)}\mathcal{W}(\gamma,x_1,\dots,x_k)\|^2$$

(here and below we omit the notation of the space in the norm if this space is clear from the context). Let $\{\mathcal{W}^{(n)}\}\subset\Omega_{k,n}$ and let $\mathcal{W}^{(n)}\to\mathcal{W}$ as $n\to\infty$ in the norm

$$\|\cdot\|_{D(\mathcal{U}_{k,n})} := (\|\cdot\|^2_{H_{k,n}}+\mathcal{U}_{k,n}(\cdot))^{1/2}.$$

Using the inequality

$$(S(\mathcal{W}^{(n)})^{1/2}-S(\mathcal{W}^{(m)})^{1/2})^2\leq S(\mathcal{W}^{(n)}-\mathcal{W}^{(m)}),$$

we conclude that $\{S(\mathcal{W}^{(n)})\}$ is a Cauchy sequence in the norm of $L^1(\Gamma_X\times X^k;\mu^{(k)})$. Let $S(\mathcal{W})$ denote its limit. Then, using the definition of $\mu^{(k)}$, we have

$$\mathcal{U}_{k,n}(\mathcal{W}) = \int_{\Gamma_X}\mu(\mathrm{d}\gamma)\int_{X^k}m(\mathrm{d}x_1)\cdots m(\mathrm{d}x_k)\rho^{(k)}(\gamma,x_1,\dots,x_k)S(\mathcal{W})(\gamma,x_1,\dots,x_k), \tag{3.16}$$

where

$$\rho^{(k)}(\gamma,x_1,\dots,x_k) := \rho(\gamma,x_1)\rho(\gamma+\varepsilon_{x_1},x_2)\cdots\rho(\gamma+\varepsilon_{x_1}+\cdots+\varepsilon_{x_{k-1}},x_k).$$

Suppose now that $\mathcal{U}_{k,n}(\mathcal{W})=0$. Then, by (ii), it follows from (3.16) that, for μ-a.e. $\gamma\in\Gamma_X$:

$$S(\mathcal{W})(\gamma,\cdot)=0\quad m^{\otimes k}\text{-a.e. on } X^k. \tag{3.17}$$

Let us fix $\gamma\in\Gamma_X$ such that (3.17) holds and let $\mathcal{O}$ be an open ball in X^k such that

$$\bar{\mathcal{O}}\subset\mathcal{X}_{k,\gamma} := \tilde{X}^k\cap(X\setminus\gamma)^k. \tag{3.18}$$

Since $\rho^{(k)}(\gamma,\cdot)$ is positive and continuous on $\bar{\mathcal{O}}$:

$$0<c_1\leq\rho^{(k)}(\gamma,\cdot)\leq c_2<\infty\quad\text{on }\mathcal{O}$$

and so L^p-convergence on $\mathcal{O}$ with respect to the measure $\sigma^{(k)}(\gamma,\mathrm{d}x_1,\dots,\mathrm{d}x_k)$ is equivalent to the same convergence with respect to the measure $m^{\otimes k}$.

Let $W_2^1(\mathcal{O})$ denote the Sobolev space consisting of all functions $f \in L^2(\mathcal{O}; m^{\otimes k})$ which are weakly differentiable and whose weak gradient $\nabla^{X^k} f \in L^2(\mathcal{O} \to T\mathcal{O}; m^{\otimes k})$.

Lemma 3.10. *We have* $\|\mathcal{W}(\gamma,\cdot)\| \in W_2^1(\mathcal{O})$ *and* $\nabla^X \|\mathcal{W}(\gamma,\cdot)\| = 0$ $m^{\otimes k}$*-a.e. on* $\mathcal{O}$.

Proof. Let us consider the classical pre-Dirichlet form on $L^2(\mathcal{O}; m^{\otimes k})$:

$$\mathcal{E}(f^{(1)}, f^{(2)}) = \int_{\mathcal{O}} \langle \nabla^{X^k} f^{(1)}(x_1,\dots,x_k), \nabla^{X^k} f^{(2)}(x_1,\dots,x_k)\rangle_{T_{(x_1,\dots,x_k)}X^k} \times m(\mathrm{d}x_1)\cdots m(\mathrm{d}x_k),$$

where $f^{(1)}, f^{(2)} \in D(\mathcal{E}) := C^1(\bar{\mathcal{O}})$. As well known, this pre-Dirichlet form is closable and let $(\mathcal{E}, D(\mathcal{E}))$ denote its closure. Then, $D(\mathcal{E}) = W_2^1(\mathcal{O})$ and

$$\mathcal{E}(f^{(1)}, f^{(2)}) = \int_{\mathcal{O}} \mathcal{S}(f^{(1)}, f^{(2)})(x_1,\dots,x_k)\, m(dx_1)\cdots m(\mathrm{d}x_k), \quad f^{(1)}, f^{(2)} \in D(\mathcal{E}),$$

where

$$\mathcal{S}(f^{(1)}, f^{(2)})(x_1,\dots,x_k) = \langle \nabla^{X^k} f^{(1)}(x_1,\dots,x_k), \nabla^{X^k} f^{(2)}(x_1,\dots,x_k)\rangle_{T_{(x_1,\dots,x_k)}X^k},$$

the gradient ∇^{X^k} being understood in the weak sense.

Hence, taking notice of (3.17), to prove this lemma, it suffices to show that the following claim is true: let $\omega : \mathcal{O} \to \wedge^n(T\mathcal{O})$ be a limit of a sequence $\{\omega_n\}$ of smooth n-forms on $\bar{\mathcal{O}}$ with respect to the norm $(\|\cdot\|^2_{L^2(\mathcal{O}\to\wedge^n(T\mathcal{O}); m^{\otimes k})} + \mathcal{G}(\cdot))^{1/2}$, where

$$\mathcal{G}(u) := \int_{\mathcal{O}} \|\nabla^{X^k} u(x_1,\dots,x_k)\|^2 m(\mathrm{d}x_1)\cdots m(\mathrm{d}x_k)$$

for a smooth form u. Then, $\|\omega\| \in D(\mathcal{E})$ and

$$\mathcal{S}(\|\omega\|) \le S(\omega) \quad m^{\otimes k}\text{-a.e. on } \mathcal{O}. \tag{3.19}$$

Here, $S(\omega)(x_1,\dots,x_k)$ is constructed analogously to the $S(W)(\gamma, x_1,\dots,x_k)$ above.

The proof of this claim is essentially the same as the proof of the fact that, for each $f \in D(\mathcal{E})$, $|f| \in D(\mathcal{E})$ and $\mathcal{S}(|f|) \le \mathcal{S}(f)$ $m^{\otimes k}$-a.e., which is why we limit ourselves to only outline it. So, first one shows by approximation that, for each fixed $\epsilon > 0$, $\sqrt{\langle\omega,\omega\rangle + \epsilon} \in D(\mathcal{E})$, and moreover, for any fixed $\epsilon, \epsilon' > 0$:

$$\begin{aligned}\mathcal{S}&(\sqrt{\langle\omega,\omega\rangle+\epsilon} - \sqrt{\langle\omega,\omega\rangle+\epsilon'})(x_1,\dots,x_k)\\ &\le \mathcal{S}(\omega)(x_1,\dots,x_k)\left\|\frac{\omega(x_1,\dots,x_k)}{\sqrt{\langle\omega,\omega\rangle+\epsilon}} - \frac{\omega(x_1,\dots,x_k)}{\sqrt{\langle\omega,\omega\rangle+\epsilon'}}\right\|^2\\ &\qquad m^{\otimes k}\text{-a.e. } (x_1,\dots,x_k) \in \mathcal{O}.\end{aligned} \tag{3.20}$$

Second, one sets $\epsilon_n \downarrow 0$ and shows using (3.20) that $\{\sqrt{\langle\omega,\omega\rangle+\epsilon_n}\}$ is a Cauchy sequence with respect to the norm $(\|\cdot\|^2_{L^2(\mathcal{O}; m^{\otimes k})} + \mathcal{E}(\cdot))^{1/2}$. The estimate (3.19) then trivially follows. Thus, the lemma is proved. □

By Lemma 3.10, it follows that $\|\mathcal{W}(\gamma,\cdot)\| = \text{const.}\ m^{\otimes k}$-a.e. on $\mathcal{O}$. Since $d \geq 2$, the set $\mathcal{X}_{k,\gamma}$ defined in (3.18) is open and connected, and therefore it can be covered by a countable number of open balls $\{\mathcal{O}_n\}$ satisfying $\bar{\mathcal{O}}_n \subset \mathcal{X}_{k,\gamma}$. Therefore, $\|\mathcal{W}(\gamma,\cdot)\| = \text{const.}\ m^{\otimes k}$-a.e. on $\mathcal{X}_{k,\gamma}$, and hence $m^{\otimes k}$-a.e. on X^k. Finally, by (3.15), $\|W\| = 0\ \mu \otimes m^{\otimes k}$-a.e. on $\Gamma_X \times X^k$. Thus, the theorem is proved.

3.3. deRham Laplacian on forms

Let $\mathcal{E}\Omega^n$ denote the subset of $\mathcal{F}\Omega^n$ consisting of all forms $W \in \mathcal{F}\Omega^n$ such that all derivatives of W are polynomially bounded, i.e., for each $k \in \mathbb{N}$ there exist $\varphi \in \mathcal{D}$, $\varphi \geq 0$, and $l \in \mathbb{N}$ (depending on W) such that

$$\|(\nabla^\Gamma)^{(k)} W(\gamma)\|^2_{(T_\gamma\Gamma_X)^{\otimes k} \otimes \wedge^n (T_\gamma \Gamma_X)} \leq \langle \varphi, \gamma\rangle^l \quad \text{for all } \gamma \in \Gamma_X \tag{3.21}$$

and additionally, for each fixed $\gamma \in \Gamma_X$ and $r \in \mathbb{N}$, the mapping

$$\begin{aligned}(X \setminus \gamma)^r \cap \tilde{X}^r \ni (x_1, \dots, x_r) \mapsto W(\gamma + \varepsilon_{x_1} + \cdots + \varepsilon_{x_r}) \\ \in \wedge^n (T_\gamma \Gamma_X \oplus T_{x_1} X \oplus \cdots \oplus T_{x_r} X)\end{aligned}$$

extends to a smooth form

$$X^r \ni (x_1, \dots, x_r) \mapsto \omega(x_1, \dots, x_r) \in \wedge^n (T_\gamma \Gamma_X \oplus T_{x_1} X \oplus \cdots \oplus T_{x_r} X)$$

(Notice that the locality of a form, together with the above condition of extension, will automatically imply the infinitely differentiability of the form.)

As easily seen, $\mathcal{D}\Omega^n$ is a subset of $\mathcal{E}\Omega^n$, and so we get the following chain of inclusions

$$\mathcal{D}\Omega^n \subset \mathcal{E}\Omega^n \subset \mathcal{F}\Omega^n.$$

We define linear operators

$$\mathbf{d}_n : \mathcal{E}\Omega^n \to \mathcal{E}\Omega^{n+1}, \quad n \in \mathbb{Z}_+, \quad \mathcal{E}\Omega^0 := \mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D}, \Gamma_X) \tag{3.22}$$

by

$$(\mathbf{d}_n W)(\gamma) := (n+1)^{1/2}\, \mathrm{AS}_{n+1}(\nabla^\Gamma W(\gamma)), \tag{3.23}$$

where

$$\mathrm{AS}_{n+1} : (T_\gamma \Gamma_X)^{\otimes(n+1)} \to \wedge^{n+1}(T_\gamma \Gamma_X) \tag{3.24}$$

is the antisymmetrization operator. (We notice that the polynomial boundedness of the form $\mathbf{d}_n W$ and its derivatives follows from the corresponding boundedness of $\nabla^\Gamma W$ and the fact that the norm of the operator (3.24) for each $\gamma \in \Gamma_X$ is equal to 1.)

Let us now consider $\mathbf{d}_n$ as an operator acting from the space $L^2_\mu \Omega^n$ into $L^2_\mu \Omega^{n+1}$. (We remark that, by the proof of Lemma 3.4, $\mathbf{d}_n W = 0$ μ-a.e. for $W \in \mathcal{E}\Omega^n$ such that $W = 0$ μ-a.e.) We denote by $\mathbf{d}_n^*$ the adjoint operator of $\mathbf{d}_n$.

Proposition 3.11. *Let (3.8) hold. Then, $\mathbf{d}_n^*$ is a densely defined operator from $L^2_\mu \Omega^{n+1}$ into $L^2_\mu \Omega^n$ with domain containing $\mathcal{E}\Omega^{n+1}$.*

Proof. It follows from (3.23) and the definition of ∇^Γ that, for any $W \in \mathcal{E}\Omega^n$ and $\gamma \in \Gamma_X$:

$$(\mathbf{d}_n W)(\gamma) = \sum_{x\in\gamma} (\mathbf{d}_{x,n} W)(\gamma), \tag{3.25}$$

where

$$(\mathbf{d}_{x,n} W)(\gamma) := (n+1)^{1/2} \mathrm{AS}_{n+1}(\nabla_x^X W(\gamma)). \tag{3.26}$$

Let $\gamma \in \Gamma_X$ and $x \in \gamma$ be fixed. Let $C^\infty(\mathcal{O}_{\gamma,x} \to \wedge^n(T_\gamma \Gamma_X))$ denote the space of all smooth sections of the Hilbert bundle (2.6). We define an operator

$$d_{x,n}^X : C^\infty(\mathcal{O}_{\gamma,x} \to \wedge^n(T_\gamma\Gamma_X)) \to C^\infty(\mathcal{O}_{\gamma,x} \to \wedge^{n+1}(T_\gamma\Gamma_X)),$$

whose action, in local coordinates on the manifold X, is given by

$$d_{x,n}^X \phi(y)\, h_1 \wedge \cdots \wedge h_n = (n+1)^{1/2} \nabla^X \phi(y) \wedge h_1 \wedge \cdots \wedge h_n, \tag{3.27}$$

$\phi \in C^\infty(\mathcal{O}_{\gamma,x} \to \mathbb{R})$, $h_k \in T_{x_k}X$, $x_k \in \gamma$, $k = 1, \dots, n$. It follows from (3.26) and (3.27) that

$$(\mathbf{d}_{x,n} W)(\gamma) = d_{x,n}^X W_x(\gamma, x). \tag{3.28}$$

Next, let $\Omega(\mathcal{O}_{\gamma,x} \to \wedge^n(T_\gamma\Gamma_X))$ denote the space of all sections of the Hilbert bundle (2.6). We define an operator

$$\delta_{x,n}^X : C^\infty(\mathcal{O}_{\gamma,x} \to \wedge^{n+1}(T_\gamma\Gamma_X)) \to \Omega(\mathcal{O}_{\gamma,x} \to \wedge^n(T_\gamma\Gamma_X))$$

setting

$$\begin{aligned} &\delta_{x,n}^X \phi(y) h_1 \wedge \cdots \wedge h_{n+1} \\ &\quad := -(n+1)^{-1/2} \sum_{i=1}^{n+1} (-1)^{i-1} \varepsilon_{x,x_i} [\langle \nabla^X \phi(y), h_i\rangle_x + \phi(y)\langle B_\mu(\gamma, y), h_i\rangle_x] h_1 \\ &\qquad \wedge \cdots \wedge \checkmark h_i \wedge \cdots \wedge h_{n+1}, \end{aligned} \tag{3.29}$$

where $\phi \in C^\infty(\mathcal{O}_{\gamma,x} \to \mathbb{R})$, $h_i \in T_{x_i}X$, $x_i \in \gamma$, $i = 1, \dots, n+1$

$$\varepsilon_{x,x_i} := \begin{cases} 1 & x = x_i, \\ 0 & \text{otherwise} \end{cases}$$

and $\checkmark h_i$ denotes the absence of h_i. We now set for $W \in \mathcal{E}\Omega^{n+1}$

$$(\boldsymbol{\delta}_{x,n}(\gamma) := \delta_{x,n}^X W_x(\gamma, x) \tag{3.30}$$

and

$$(\boldsymbol{\delta}_n W)(\gamma) := \sum_{x\in\gamma} (\boldsymbol{\delta}_{x,n} W)(\gamma) \tag{3.31}$$

(Notice that the sum on the right hand side of (3.31) is actually finite.)

Let us show that, for any $W \in \mathcal{E}\Omega^{n+1}$, we have $\boldsymbol{\delta}_n W \in L^2_\mu\Omega^n$, where $L^2_\mu\Omega^0 := L^2(\Gamma_X;\mu)$. We choose any $\Lambda \in \mathcal{O}_c(X)$ such that $W(\gamma) = W(\gamma_{\Lambda'})$ for some compact $\Lambda' \subset \Lambda$. Then, by (3.31):

$$\int_{\Gamma_X} \|(\boldsymbol{\delta}_n W)(\gamma)\|^2_{\wedge^n(T_\gamma\Gamma_X)}\mu(\mathrm{d}\gamma) = \int_{\Gamma_X} \left\| \sum_{x\in\gamma_\Lambda} (\boldsymbol{\delta}_{x,n} W)(\gamma)\right\|^2_{\wedge^n(T_\gamma\Gamma_X)} \mu(\mathrm{d}\gamma)$$
$$\leq \int_{\Gamma_X} \left(\sum_{x\in\gamma_\Lambda} \|(\boldsymbol{\delta}_{x,n} W)(\gamma)\|_{\wedge^n(T_\gamma\Gamma_X)}\right)^2 \mu(\mathrm{d}\gamma). \tag{3.32}$$

Using (3.21), (3.29) and (3.30), it is not hard to show that there exist $\varphi \in C_0(X)$, $\varphi \geq 0$, and $k \in \mathbb{N}$ (independent of γ and x) such that

$$\|(\boldsymbol{\delta}_{x,n} W)(\gamma)\|_{\wedge^n(T_\gamma\Gamma_X)} \leq \langle\varphi,\gamma\rangle^k + (n+1)^{1/2}|B_\mu(\gamma,x)|_x\|W(\gamma)\|_{\wedge^{n+1}(T_\gamma\Gamma_X)}. \tag{3.33}$$

Analogously to the proof of (3.13), we get from (2.7), (2.8), (3.8), (3.32) and (3.33) that $\boldsymbol{\delta}_n W \in L^2_\mu\Omega^n$.

Let $W^{(1)}, W^{(2)} \in \mathcal{E}\Omega^n$ and let $\Lambda \in \mathcal{O}_c(X)$ be such that, for some compact $\Lambda' \subset \Lambda$ $W^{(i)}(\gamma) = W^{(i)}(\gamma_{\Lambda'})$, $i = 1, 2$, for all $\gamma \in \Gamma_X$. Then, by (2.10), (3.25), (3.27) and (3.28), we get using the notations of Section 2

$$\begin{aligned}
&\int_{\Gamma_X} \langle \mathbf{d}_n W^{(1)}(\gamma), W^{(2)}(\gamma)\rangle_{\wedge^{n+1}(T_\gamma\Gamma_X)}\mu(\mathrm{d}\gamma)\\
&= \int_{\Gamma_X}\mu(\mathrm{d}\gamma)\int_\Lambda \gamma(\mathrm{d}x)\langle(\mathbf{d}_{x,n}W^{(1)})(\gamma), W^{(2)}(\gamma)\rangle_{\wedge^{n+1}(T_\gamma\Gamma_X)}\\
&= \int_{\Gamma_X}\mu(\mathrm{d}\gamma)\int_\Lambda \sigma(\gamma,\mathrm{d}x)\langle(\mathbf{d}_{x,n}W^{(1)})(\gamma+\varepsilon_x), W^{(2)}(\gamma+\varepsilon_x)\rangle_{\wedge^{n+1}(T_{\gamma+\varepsilon_x}\Gamma_X)}\\
&= \int_{\Gamma_X}\mu(\mathrm{d}\gamma)\int_\Lambda \sigma(\gamma,\mathrm{d}x)\sum_{k=1}^n \sum_{\substack{\{x_1,\dots,x_k\}\subset\gamma\cup\{x\}\\ x\in\{x_1,\dots,x_k\}}} \langle(\mathbf{d}_{x,n}W^{(1)})_k(\gamma+\varepsilon_x, x_1,\dots,x_k),\\
&\qquad W^{(2)}_k(\gamma+\varepsilon_x,x_1,\dots,x_k)\rangle_{\wedge^{n+1}(T_{\gamma+\varepsilon_x}\Gamma_X)}\\
&= \int_{\Gamma_X}\mu(\mathrm{d}\gamma)\sum_{k=1}^n\sum_{\{x_1,\dots,x_{k-1}\}\subset\gamma}\int_\Lambda \sigma(\gamma,\mathrm{d}x)\langle(\mathbf{d}_{x,n}W^{(1)})_k(\gamma+\varepsilon_x,x,x_1,\dots,x_{k-1}),\\
&\qquad W^{(2)}_k(\gamma+\varepsilon_x,x,x_1,\dots,x_{k-1})\rangle_{\wedge^{n+1}(T_{\gamma+\varepsilon_x}\Gamma_X)}.
\end{aligned} \tag{3.34}$$

It follows from the definition of $\mathcal{E}\Omega^n$ that, for a fixed $\gamma \in \Gamma_X$ and $\{x_1,\dots,x_{k-1}\} \subset \gamma$, $W^{(2)}_k(\gamma,\cdot,x_1,\dots,x_{k-1})$ extends to a smooth form

$$X \ni x \mapsto W_k^{(2)}(\gamma, x, x_1, \dots, x_{k-1}) \in \bigoplus_{\substack{1\le l_1,\dots,l_k\le d\\ l_1+\dots+l_k=n}} (T_xX)^{\wedge l_1} \wedge (T_{x_1}X)^{\wedge l_2} \wedge \cdots$$
$$\wedge (T_{x_{k-1}}X)^{\wedge l_k} \subset (T_xX \oplus T_{x_1}X \oplus T_{x_{k-1}}X)^{\wedge n}.$$

Since $W^{(1)}(\gamma + \varepsilon_\bullet)$ also extends to a smooth form on X, we can carry out an integration by parts in the x variable in (3.34). Thus, by using (3.29)–(3.31) and (2.10), we continue (3.34) as follows:

$$\begin{aligned}
&= \int_{\Gamma_X} \mu(\mathrm{d}\gamma) \sum_{k=1}^{n} \sum_{\{x_1,\dots,x_{k-1}\}\subset\gamma} \\
&\quad \times \int_\Lambda \sigma(\gamma, \mathrm{d}x) \langle W^{(1)}(\gamma+\varepsilon_x), \delta^X_{x,n} W_k^{(2)}(\gamma+\varepsilon_x, x, x_1, \dots, x_{k-1})\rangle_{\wedge^n(T_{\gamma+\varepsilon_x}\Gamma_X)} \\
&= \int_{\Gamma_X} \mu(\mathrm{d}\gamma) \int_\Lambda \sigma(\gamma, \mathrm{d}x) \langle W^{(1)}(\gamma+\varepsilon_x), \delta^X_{x,n} W_x^{(2)}(\gamma+\varepsilon_x, x)\rangle_{\wedge^n(T_{\gamma+\varepsilon_x}\Gamma_X)} \\
&= \int_{\Gamma_X} \mu(\mathrm{d}\gamma) \int_\Lambda \sigma(\gamma, \mathrm{d}x) \langle W^{(1)}(\gamma+\varepsilon_x), (\boldsymbol{\delta}_{x,n} W^{(2)})(\gamma+\varepsilon_x)\rangle_{\wedge^n(T_{\gamma+\varepsilon_x}\Gamma_X)} \\
&= \int_{\Gamma_X} \mu(\mathrm{d}\gamma) \int_\Lambda \gamma(\mathrm{d}x)\, \langle W^{(1)}(\gamma), (\boldsymbol{\delta}_{x,n} W^{(2)})(\gamma)\rangle_{\wedge^n(T_\gamma\Gamma_X)} \\
&= \int_{\Gamma_X} \langle W^{(1)}(\gamma), (\boldsymbol{\delta}_n W^{(2)})(\gamma)\rangle_{\wedge^n(T_\gamma\Gamma_X)} \mu(\mathrm{d}\gamma).
\end{aligned}$$

Hence, $\mathcal{F}\Omega^{n+1} \subset D(\mathbf{d}^*_{\mu,n})$ and $\mathbf{d}^*_n \rightharpoonup \mathcal{E}\Omega^{n+1} = \boldsymbol{\delta}_{\mu,n}$. □

Corollary 3.12. *The operator* $\mathbf{d}_n : L^2_\mu\Omega^n \to L^2_\mu\Omega^{n+1}$ *is closable.*

We denote by $\bar{\mathbf{d}}_n$ the closure of $\mathbf{d}_n$. The space $Z^n := \operatorname{Ker} \bar{\mathbf{d}}_n$ is then a closed subspace of $L^2_\pi\Omega^n$. Let B^n denote the closure in $L^2_\pi\Omega^n$ of the subspace $\operatorname{Im} \mathbf{d}_{n-1}$ (of course, B^n =the closure of $\operatorname{Im} \bar{\mathbf{d}}_{n-1}$).

We obviously have $\mathbf{d}_n\mathbf{d}_{n-1} = 0$, which implies

$$\operatorname{Im} \mathbf{d}_{n-1} \subset \operatorname{Ker} \mathbf{d}_n \subset Z^n.$$

Hence $B^n \subset Z^n$ and

$$\bar{\mathbf{d}}_n \bar{\mathbf{d}}_{n-1} = 0. \tag{3.35}$$

Thus, we have the infinite complex

$$\cdots \xrightarrow{\mathbf{d}_{n-1}} \mathcal{E}\Omega^n \xrightarrow{\mathbf{d}_n} \mathcal{E}\Omega^{n+1} \xrightarrow{\mathbf{d}_{n+1}} \cdots$$

and the associated Hilbert complex

$$\cdots \xrightarrow{\bar{\mathbf{d}}_{n-1}} L^2_\pi\Omega^n \xrightarrow{\bar{\mathbf{d}}_n} L^2_\pi\Omega^{n+1} \xrightarrow{\bar{\mathbf{d}}_{n+1}} \cdots . \tag{3.36}$$

We set in a standard way

$$\mathcal{H}^n_\mu = Z^n/B^n, \quad n \in \mathbb{N}.$$

For $n \in \mathbb{N}$, we define a bilinear form $\mathcal{E}^{\mathrm{R}}_{\mu,n}$ on $L^2_\mu \Omega^n$ by

$$\mathcal{E}^{\mathrm{R}}_{\mu,n}(W^{(1)}, W^{(2)}) := \int_{\Gamma_X} [\langle \bar{\mathbf{d}}_n W^{(1)}(\gamma), \bar{\mathbf{d}}_n W^{(2)}(\gamma)\rangle_{\wedge^{n+1}(T_\gamma \Gamma_X)} + \langle \mathbf{d}^*_{n-1} W^{(1)}(\gamma), \mathbf{d}^*_{n-1} W^{(2)}(\gamma)\rangle_{\wedge^{n-1}(T_\gamma \Gamma_X)}]\mu(\mathrm{d}\gamma), \tag{3.37}$$

where $W^{(1)}, W^{(2)} \in D(\mathcal{E}^{\mathrm{R}}_{\mu,n}) := D(\bar{\mathbf{d}}_n) \cap D(\mathbf{d}^*_{n-1})$. This form is evidently closed, and let $(\mathbf{H}^{\mathrm{R}}_{\mu,n}, D(\mathbf{H}^{\mathrm{R}}_{\mu,n}))$ denote its generator. This operator will be called *the Hodge–deRham Laplacian of the measure* μ.

The following proposition reflects a quite standard fact in the theory of L^2-cohomologies.

Proposition 3.13. *The natural isomorphism between $\mathcal{H}^n_\mu$ and the orthogonal complement of B^n to Z^n is the isomorphism of the Hilbert spaces*

$$\mathcal{H}^n_\mu \simeq \operatorname{Ker} \mathbf{H}^{\mathrm{R}}_{\mu,n}. \tag{3.38}$$

Proof. Using [12, Proposition A.1], we conclude from Proposition 3.11 and formula (3.35) that

$$L^2_\mu \Omega^n = \operatorname{Ker} \mathbf{H}^{\mathrm{R}}_{\mu,n} \oplus \overline{\operatorname{Im} \mathbf{d}_{n-1}} \oplus \overline{\operatorname{Im} \mathbf{d}^*_n} \tag{3.39}$$

(the weak Hodge–deRham decomposition). For the closed operator $\bar{\mathbf{d}}_n$ we have the standard decomposition

$$L^2_\mu \Omega^n = \operatorname{Ker} \bar{\mathbf{d}}_n \oplus \overline{\operatorname{Im} \mathbf{d}^*_n},$$

which together with (3.39) implies the result. □

We do not know a priori whether the domain $D(\mathbf{H}^{\mathrm{R}}_{\mu,n})$ contains $\mathcal{D}\Omega^n$, however the following theorem gives a sufficient condition for this.

Theorem 3.14. *Let us suppose that*

(i) *For μ-a.e. $\gamma \in \Gamma_X$, $\rho(\gamma, x) > 0$ for all $x \in X \setminus \gamma$ and the function $\rho(\gamma, \cdot)$ is continuous on X.*
(ii) *For μ-a.e. $\gamma \in \Gamma_X$, $\rho(\gamma, \cdot)$ is two times differentiable on $X \setminus \gamma$ and $\nabla^X \rho(\gamma, \cdot)$ extends to a continuous form on X.*
(iii) *For $\mu \otimes m$-a.e. $(\gamma, x) \in \Gamma_X \times X$, $y \mapsto \nabla^X_x \rho(\gamma + \varepsilon_y, x) \in T_x X$ is differentiable on $X \setminus (\gamma \cup \{x\})$ and*

$$X \setminus (\gamma \cup \{x\}) \ni y \mapsto \frac{\rho(\gamma + \varepsilon_x, y)}{\rho(\gamma + \varepsilon_y, x)} \nabla^X_x \rho(\gamma + \varepsilon_y, x) \in T_x X$$

extends to a continuous mapping on X.

(iv) (3.8) *holds, and furthermore*

$$\forall\,\Lambda\in\mathcal{O}_{\rm c}(X)\,\exists\varepsilon>0:\int_{\Gamma_X}\left(\sum_{y\in\gamma}\sum_{x\in\gamma_\Lambda}\|\nabla_y^X B_\mu(\gamma,x)\|_{T_yX\otimes T_xX}\right)^{2+\varepsilon}\mu(\mathrm{d}\gamma)<\infty. \tag{3.40}$$

Then, $\mathcal{D}\Omega^n\subset D(\mathbf{H}^{\rm R}_{\mu,n})$ *and*

$$\mathbf{H}^{\rm R}_{\mu,n}\rightharpoonup\mathcal{D}\Omega^n=\mathbf{d}_{n-1}\mathbf{d}^*_{n-1}+\mathbf{d}^*_n\mathbf{d}_n.$$

Proof. Since by Proposition 3.11 $\mathbf{d}_n\mathcal{D}\Omega^n\subset\mathcal{E}\Omega^{n+1}\subset D(\mathbf{d}_n^*)$, to prove the theorem we have to show that $\mathbf{d}^*_{n-1}\mathcal{D}\Omega^n\subset D(\mathbf{d}_{n-1})$, i.e., for arbitrary $W^{(1)},W^{(2)}\in\mathcal{D}\Omega^n$, there exits $V\in L^2_\mu\Omega^n$ such that

$$\int_{\Gamma_X}\langle\mathbf{d}^*_{n-1}W^{(1)}(\gamma),\mathbf{d}^*_{n-1}W^{(2)}(\gamma)\rangle_{\wedge^n(T_\gamma\Gamma_X)}\,\mu(\mathrm{d}\gamma)=\int_{\Gamma_X}\langle V(\gamma),W^{(2)}(\gamma)\rangle_{\wedge^n(T_\gamma\Gamma_X)}\mu(\mathrm{d}\gamma). \tag{3.41}$$

We choose any $\Lambda\in\mathcal{O}_{\rm c}(X)$ such that, for some compact $\Lambda'\subset\Lambda$, $W^{(i)}(\gamma)=W^{(i)}(\gamma_{\Lambda'})$, $i=1,2$, for all $\gamma\in\Gamma_X$. It follows from the proof of Proposition 3.11 and Lemma 2.1 that

$$\begin{aligned}
&\int_{\Gamma_X}\langle\mathbf{d}^*_{n-1}W^{(1)}(\gamma),\mathbf{d}^*_{n-1}W^{(2)}(\gamma)\rangle_{\wedge^{n-1}(T_\gamma\Gamma_X)}\,\mu(\mathrm{d}\gamma)\\
&\quad=\int_{\Gamma_X}\sum_{x,y\in\gamma_\Lambda}\langle\boldsymbol{\delta}_{x,n}W^{(1)}(\gamma),\boldsymbol{\delta}_{y,n}W^{(2)}(\gamma)\rangle_{\wedge^{n-1}(T_\gamma\Gamma_X)}\mu(\mathrm{d}\gamma)\\
&\quad=\int_{\Gamma_X}\mu(\mathrm{d}\gamma)\int_\Lambda\sigma(\gamma,\mathrm{d}x)\langle\boldsymbol{\delta}_{x,n}W^{(1)}(\gamma+\varepsilon_x),\boldsymbol{\delta}_{x,n}W^{(2)}(\gamma+\varepsilon_x)\rangle_{\wedge^{n-1}(T_{\gamma+\varepsilon_x}\Gamma_X)}\\
&\qquad+\int_{\Gamma_X}\mu(\mathrm{d}\gamma)\int_\Lambda\sigma(\gamma,\mathrm{d}x)\int_\Lambda\sigma(\gamma+\varepsilon_x,\mathrm{d}y)\langle\boldsymbol{\delta}_{x,n}W^{(1)}(\gamma+\varepsilon_x+\varepsilon_y),\\
&\qquad\boldsymbol{\delta}_{y,n}W^{(2)}(\gamma+\varepsilon_x+\varepsilon_y)\rangle_{\wedge^{n-1}(T_{\gamma+\varepsilon_x+\varepsilon_y}\Gamma_X)}.
\end{aligned} \tag{3.42}$$

Due to conditions (i)–(iii), we can see that, for μ-a.e. $\gamma\in\Gamma_X$ and $x\in X\setminus\gamma$:

$$\mathbf{d}_{x,n}\boldsymbol{\delta}_{x,n}W^{(1)}(\gamma+\varepsilon_x)\in\wedge^n(T_{\gamma+\varepsilon_x})$$

and for $\mu\otimes m$-a.e. $(\gamma,x)\in\Gamma_X\times X$ and $y\in X\setminus(\gamma\cup\{x\})$:

$$\mathbf{d}_{y,n}\boldsymbol{\delta}_{x,n}W^{(1)}(\gamma+\varepsilon_x+\varepsilon_y)\in\wedge^n(T_{\gamma+\varepsilon_x+\varepsilon_y}\Gamma_X),$$

using formulas (3.27) and (3.28) for the definition of $\mathbf{d}_{x,n}$, $x\in X$. Moreover, by virtue of (i) and (ii), the integration by parts yields, for μ-a.e. $\gamma\in\Gamma_X$

$$\begin{aligned}
&\int_\Lambda\sigma(\gamma,\mathrm{d}x)\langle\boldsymbol{\delta}_{x,n}W^{(1)}(\gamma+\varepsilon_x),\boldsymbol{\delta}_{x,n}W^{(2)}(\gamma+\varepsilon_x)\rangle_{\wedge^{n-1}(T_{\gamma+\varepsilon_x}\Gamma_X)}\\
&\quad=\int_\Lambda\sigma(\gamma,\mathrm{d}x)\langle\mathbf{d}_{x,n}\boldsymbol{\delta}_{x,n}W^{(1)}(\gamma+\varepsilon_x),W^{(2)}(\gamma+\varepsilon_x)\rangle_{\wedge^n(T_{\gamma+\varepsilon_x}\Gamma_X)}
\end{aligned} \tag{3.43}$$

and analogously, using (i) and (iii), we get, for $\mu \otimes m$-a.e. $(\gamma, x) \in \Gamma_X \times X$:

$$\int_\Lambda \sigma(\gamma + \varepsilon_x, \mathrm{d}y)\, \langle \boldsymbol{\delta}_{x,n} W^{(1)}(\gamma + \varepsilon_x + \varepsilon_y), \boldsymbol{\delta}_{y,n} W^{(2)}(\gamma + \varepsilon_x + \varepsilon_y)\rangle_{\wedge^{n-1}(T_{\gamma+\varepsilon_x+\varepsilon_y}\Gamma_X)} = \int_\Lambda \sigma(\gamma + \varepsilon_x, \mathrm{d}y) \langle \mathbf{d}_{y,n} \boldsymbol{\delta}_{x,n} W^{(1)}(\gamma + \varepsilon_x + \varepsilon_y), W^{(2)}(\gamma + \varepsilon_x + \varepsilon_y)\rangle_{\wedge^{n}(T_{\gamma+\varepsilon_x+\varepsilon_y}\Gamma_X)}. \tag{3.44}$$

Suppose that

$$\int_{\Gamma_X} \left(\sum_{x,y\in\gamma} \| \mathbf{d}_{y,n} \boldsymbol{\delta}_{x,n} W^{(1)}(\gamma)\|_{\wedge^n(T_\gamma \Gamma_X)} \right)^2 \mu(\mathrm{d}\gamma) < \infty \tag{3.45}$$

so that

$$V(\gamma) := \sum_{x,y\in\gamma} \mathbf{d}_{y,n} \boldsymbol{\delta}_{x,n} W^{(1)}(\gamma) \in \wedge^n(T_\gamma \Gamma_X)$$

is well defined for μ-a.a. $\gamma \in \Gamma_X$, and moreover $V \in L^2_\mu \Omega^n$. Then, by Lemma 2.1 and (3.43)–(3.45), we continue (3.42) as follows:

$$\begin{aligned} &= \int_{\Gamma_X} \mu(\mathrm{d}\gamma) \int_\Lambda \sigma(\gamma, \mathrm{d}x) \langle \mathbf{d}_{x,n} \boldsymbol{\delta}_{x,n} W^{(1)}(\gamma + \varepsilon_x), W^{(2)}(\gamma + \varepsilon_x)\rangle_{\wedge^n(T_{\gamma+\varepsilon_x}\Gamma_X)} \\ &\quad + \int_{\Gamma_X} \mu(\mathrm{d}\gamma) \int_\Lambda \sigma(\gamma, \mathrm{d}x) \int_\Lambda \sigma(\gamma + \varepsilon_x, \mathrm{d}y) \langle \mathbf{d}_{y,n} \boldsymbol{\delta}_{x,n} W^{(1)}(\gamma + \varepsilon_x + \varepsilon_y), \\ &\quad W^{(2)}(\gamma + \varepsilon_x + \varepsilon_y)\rangle_{\wedge^n(T_{\gamma+\varepsilon_x+\varepsilon_y}\Gamma_X)} \\ &= \int_{\Gamma_X} \langle V_\Lambda(\gamma), W^{(2)}(\gamma)\rangle_{\wedge^n(T_\gamma\Gamma_X)} \mu(\mathrm{d}\gamma), \end{aligned} \tag{3.46}$$

where

$$V_\Lambda(\gamma) := \sum_{x,y\in\gamma_\Lambda} \mathbf{d}_{y,n} \boldsymbol{\delta}_{x,n} W^{(1)}(\gamma). \tag{3.47}$$

Since $V_\Lambda(\gamma) \to V(\gamma)$ as $\Lambda \to X$ for μ-a.e. $\gamma \in \Gamma_X$, by the majorized convergence theorem, we conclude from (3.42), (3.46) and (3.47) that

$$\int_{\Gamma_X} \langle \mathbf{d}^*_{n-1} W^{(1)}(\gamma), \mathbf{d}^*_{n-1} W^{(2)}(\gamma)\rangle_{\wedge^{n-1}(T_\gamma\Gamma_X)} \mu(\mathrm{d}\gamma) = \int_{\Gamma_X} \langle V(\gamma), W^{(2)}(\gamma)\rangle_{\wedge^n(T_\gamma\Gamma_X)} \mu(\mathrm{d}\gamma).$$

Thus, it remains to show that (3.45) does indeed hold. Let $\tilde\Lambda \in \mathcal{O}_c(X)$ be such that, for some compact $\Lambda' \subset \tilde\Lambda$, $W^{(1)}(\gamma) = W^{(1)}(\gamma_{\Lambda'})$ for all $\gamma \in \Gamma_X$ ($\tilde\Lambda$ being now independent of $W^{(2)}$). Since $\boldsymbol{\delta}_{x,n} W^{(1)}(\gamma) = 0$ for all $x \in \gamma_{\tilde\Lambda^c}$, we get

$$
\begin{aligned}
&\int_{\Gamma_X}\left(\sum_{x,y\in\gamma}\|\mathbf{d}_{y,n}\boldsymbol{\delta}_{x,n}W^{(1)}(\gamma)\|_{\wedge^n(T_\gamma\Gamma_X)}\right)^2\mu(\mathrm{d}\gamma)\\
&\quad\le 2\int_{\Gamma_X}\left(\sum_{x,y\in\gamma_{\tilde\Lambda}}\|\mathbf{d}_{y,n}\boldsymbol{\delta}W^{(1)}(\gamma)\|_{\wedge^n(T_\gamma\Gamma_X)}\right)^2\mu(\mathrm{d}\gamma)\\
&\qquad+2\int_{\Gamma_X}\left(\sum_{y\in\gamma_{\tilde\Lambda^{\mathrm{c}}}}\sum_{x\in\gamma_{\tilde\Lambda}}\|\mathbf{d}_{y,n}\boldsymbol{\delta}_{x,n}W^{(1)}(\gamma)\|_{\wedge^n(T_\gamma\Gamma_X)}\right)^2\mu(\mathrm{d}\gamma).
\end{aligned}\tag{3.48}
$$

Analogously to (3.33), we conclude from (3.27)–(3.30) the existence of $\varphi\in C_0(X)$, $\varphi\ge 0$, and $k\in\mathbb{N}$ (independent of γ, x, and y) such that

$$
\|\mathbf{d}_{y,n}\boldsymbol{\delta}_{x,n}W^{(1)}(\gamma)\|_{\wedge^n(T_\gamma\Gamma_X)}\le\langle\varphi,\gamma\rangle^k(1+|B_\mu(\gamma,x)|_x+\|\nabla^X_y B_\mu(\gamma,x)\|_{T_yX\otimes T_xX})\tag{3.49}
$$

for $x, y\in\gamma_{\tilde\Lambda}$, and

$$
\|\mathbf{d}_{y,n}\boldsymbol{\delta}_{x,n}W^{(1)}(\gamma)\|_{\wedge^n(T_\gamma\Gamma_X)}\le\langle\varphi,\gamma\rangle^k\|\nabla^X_y B_\mu(\gamma,x)\|_{T_yX\otimes T_xX}\tag{3.50}
$$

for $y\in\gamma_{\tilde\Lambda^c}$ and $x\in\gamma_{\tilde\Lambda}$.

Thus, the finiteness of the right hand side of (3.48) can easily be deduced from (2.8), (3.8), (3.40), (3.49) and (3.50), and the Schwarz inequality. □

Corollary 3.15. *Let the conditions of Theorem 3.14 be satisfied. Then, for each $W\in\mathcal{D}\Omega^n$ and μ-a.e. $\gamma\in\Gamma_X$*

$$
\sum_{x,y\in\gamma}(\|\boldsymbol{\delta}_{x,n}\mathbf{d}_{y,n}W(\gamma)\|_{\wedge^n(T_\gamma\Gamma_X)}+\|\mathbf{d}_{y,n}\boldsymbol{\delta}_{x,n}W(\gamma)\||_{\wedge^n(T_\gamma\Gamma_X)})<\infty\tag{3.51}
$$

and the action of the operator $\mathbf{H}^{\mathrm{R}}_{\mu,n}$ can be represented in the form

$$
\mathbf{H}^{\mathrm{R}}_{\mu,n}W(\gamma)=\sum_{x,y\in\gamma}(\boldsymbol{\delta}_{x,n}\mathbf{d}_{y,n}+\mathbf{d}_{y,n}\boldsymbol{\delta}_{x,n})W(\gamma),\quad \mu\text{-a.e. }\gamma\in\Gamma_X.
$$

3.4. Weitzenböck formula

In this section, we will derive a Weitzenböck type formula, which gives a relation between the Bochner Laplacian $\mathbf{H}^{\mathrm{B}}_{\mu,n}$ and the deRham Laplacian $\mathbf{H}^{\mathrm{R}}_{\mu,n}$. In what follows, we will suppose that the conditions of Theorem 3.14 are satisfied.

For each $V(\gamma)\in T_\gamma\Gamma_X$, $\gamma\in\Gamma_X$, we define an annihilation operator

$$
a_n(V(\gamma)):\wedge^n(T_\gamma\Gamma_X)\to\wedge^{n-1}(T_\gamma\Gamma_X)
$$

and a creation operator

$$
a_n^*(V(\gamma)):\wedge^{n-1}(T_\gamma\Gamma_X)\to\wedge^n(T_\gamma\Gamma_X)
$$

as follows:

$$a_n(V(\gamma))W_n(\gamma) := \sqrt{n}\langle V(\gamma), W_n(\gamma)\rangle_\gamma, \quad W_n(\gamma) \in \wedge^n(T_\gamma\Gamma_X),$$
$$a_n^*(V(\gamma))W_{n-1}(\gamma) := \sqrt{n}V(\gamma) \wedge W_{n-1}(\gamma), \quad W_{n-1}(\gamma) \in \wedge^{n-1}(T_\gamma\Gamma_X)$$

(the pairing in the expression $\langle V(\gamma), W_n(\gamma)\rangle_\gamma$ is carried out in the first "variable," so that $a_n^*(V(\gamma))$ becomes the adjoint of $a_n(V(\gamma))$).

Now, for a fixed $\gamma \in \Gamma_X$, we define an operator $R_n(\gamma)$ in $\wedge^n(T_\gamma\Gamma_X)$ as follows:

$$R_n(\gamma) := \sum_{x\in\gamma} R(\gamma, x), \qquad D(R_n(\gamma)) := \wedge_0^n(T_\gamma\Gamma_X),$$

$$R_n(\gamma, x) := \sum_{i,j,k,l=1}^{d} R_{i,j,k,l}(x)a_n^*(e_i)a_n(e_j)a_n^*(e_k)a_n(e_l).$$

Here, $\{e_j\}_{j=1}^d$ is a fixed orthonormal basis in the space T_xX considered as a subspace of $T_\gamma\Gamma_X$, $\wedge_0^n(T_\gamma\Gamma_X)$ consists of all $W_n(\gamma) \in \wedge^n(T_\gamma\Gamma_X)$ having only a finite number of nonzero coordinates in the direct sum expansion (2.16), and R_{ijkl} is the curvature tensor on X.

Next, let $A(\gamma) \in (T_{\gamma,\infty}\Gamma_X)^{\otimes 2}$, so that $A(\gamma) = (A(\gamma, x, y))_{x,y\in\gamma}$, where $A(\gamma, x, y) \in T_yX \otimes T_xX$. We realize $A(\gamma)$ as a linear operator acting from $T_{\gamma,0}\Gamma_X$ into $T_{\gamma,\infty}\Gamma_X$ by setting

$$T_{\gamma,0}\Gamma_X \ni V(\gamma) = (V(\gamma, x))_{x\in\gamma} \mapsto A(\gamma)V(\gamma)$$
$$:= \left(\sum_{x\in\gamma}\langle A(\gamma, x, y), V(\gamma, x)\rangle_x\right)_{y\in\gamma} \in T_{\gamma,\infty}\Gamma_X.$$

If we additionally suppose that, for any $\Lambda \in \mathcal{O}_c(X)$:

$$\sum_{y\in\gamma}\left(\sum_{x\in\gamma_\Lambda}\|A(\gamma, x, y)\|_{T_yX\otimes T_xX}\right)^2 < \infty,$$

then, as easily seen, $A(\gamma)$ is indeed an operator acting from $T_{\gamma,0}\Gamma_X$ into $T_\gamma\Gamma_X$. In the latter case, we define a linear operator $A(\gamma)^{\wedge n}$ in $\wedge^n(T_\gamma\Gamma_X)$ with domain $D(A(\gamma)^{\wedge n}) := \wedge_0^n(T_\gamma\Gamma_X)$ as follows:

$$A(\gamma)^{\wedge n} := A(\gamma) \otimes \mathbf{1} \cdots \otimes \mathbf{1} + \mathbf{1} \otimes A(\gamma) \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} + \cdots + \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes A(\gamma).$$

We set

$$B'_\mu(\gamma) = (B'_\mu(\gamma, x, y))_{x,y\in\gamma} \in (T_{\gamma,\infty}\Gamma_X)^{\otimes 2}, \quad B'_\mu(\gamma, x, y) := \nabla_y^X B_\mu(\gamma, x).$$

It follows from (3.40) that, for μ-a.e. $\gamma \in \Gamma_X$:

$$\sum_{y\in\gamma}\left(\sum_{x\in\gamma_\Lambda}\|B'_\mu(\gamma, x, y)\|_{T_yX\otimes T_xX}\right)^2 \le \left(\sum_{y\in\gamma}\sum_{x\in\gamma_\Lambda}\|B'_\mu(\gamma, x, y)\|_{T_yX\otimes T_xX}\right)^2 < \infty.$$

Therefore, the operator $B'_\mu(\gamma)^{\wedge n}: \wedge_0^n(T_\gamma\Gamma_X) \to \wedge^n(T_\gamma\Gamma_X)$ is well defined for μ-a.e. $\gamma\in\Gamma_X$.

Theorem 3.16. *Let the conditions of Theorem 3.14 be satisfied. Then, we have on* $\mathcal{D}\Omega^n$:

$$\mathbf{H}^{\mathrm{R}}_{\mu,n}W(\gamma) = \mathbf{H}^{\mathrm{B}}_{\mu,n} + R_n(\gamma)W(\gamma) - B'_\mu(\gamma)^{\wedge n}W(\gamma), \quad \mu\text{-a.e. } \gamma\in\Gamma_X.$$

Proof. We fix any $W^{(1)}\in\mathcal{D}\Omega^n$. By Corollary 3.15, we have

$$\begin{aligned}\mathbf{H}^{\mathrm{R}}_{\mu,n}W^{(1)}(\gamma) = {} & \sum_{x\in\gamma}(\boldsymbol{\delta}_{\mu,x,n}\mathbf{d}_{x,n} + \mathbf{d}_{x,n}\boldsymbol{\delta}_{\mu,x,n})W^{(1)}(\gamma)\\ & + \sum_{x,y\in\gamma, x\neq y}(\boldsymbol{\delta}_{\mu,x,n}\mathbf{d}_{y,n} + \mathbf{d}_{y,n}\boldsymbol{\delta}_{\mu,x,n})W^{(1)}(\gamma).\end{aligned} \tag{3.52}$$

By (2.10) and (3.51), we get for any $W^{(2)}\in\mathcal{D}\Omega^n$

$$\begin{aligned}&\int_{\Gamma_X}\left\langle\sum_{x\in\gamma}(\boldsymbol{\delta}_{\mu,x,n}\mathbf{d}_{x,n} + \mathbf{d}_{x,n}\boldsymbol{\delta}_{\mu,x,n})W^{(1)}(\gamma), W^{(2)}(\gamma)\right\rangle_{\wedge^n(T_\gamma\Gamma_X)}\mu(\mathrm{d}\gamma)\\ &= \int_{\Gamma_X}\sum_{x\in\gamma}\Big\langle(\boldsymbol{\delta}_{\mu,x,n}\mathbf{d}_{x,n} + \mathbf{d}_{x,n}\boldsymbol{\delta}_{\mu,x,n})W^{(1)}(\gamma), W^{(2)}(\gamma)\Big\rangle_{\wedge^n(T_\gamma\Gamma_X)}\mu(\mathrm{d}\gamma)\\ &= \int_{\Gamma_X}\mu(\mathrm{d}\gamma)\int_X\sigma(\gamma,\mathrm{d}x)\langle(\boldsymbol{\delta}_{\mu,x,n}\mathbf{d}_{x,n} + \mathbf{d}_{x,n}\boldsymbol{\delta}_{\mu,x,n})W^{(1)}(\gamma+\varepsilon_x),\\ &\quad W^{(2)}(\gamma+\varepsilon_x)\rangle_{\wedge^n(T_{\gamma+\varepsilon_x}\Gamma_X)}.\end{aligned} \tag{3.53}$$

By a slight modification of the proof of the Weitzenböck formula on the manifold X (see, e.g. [19]), we get for a fixed $\gamma\in\Gamma_X$

$$\begin{aligned}&\int_X\sigma(\gamma,\mathrm{d}x)\langle(\boldsymbol{\delta}_{\mu,x,n}\mathbf{d}_{x,n} + \mathbf{d}_{x,n}\boldsymbol{\delta}_{\mu,x,n})W^{(1)}(\gamma+\varepsilon_x), W^{(2)}(\gamma+\varepsilon_x)\rangle_{\wedge^n(T_{\gamma+\varepsilon_x}\Gamma_X)}\\ &= \int_X\sigma(\gamma,\mathrm{d}x)\langle-\Delta_x^X W^{(1)}(\gamma+\varepsilon_x) - \langle\nabla_x^X W^{(1)}(\gamma+\varepsilon_x), \beta_\sigma(\gamma,x)\rangle_x\\ &\quad + R_n(\gamma+\varepsilon_x,x)W^{(1)}(\gamma+\varepsilon_x) - (\nabla_x^X\beta_\sigma(\gamma,x))^{\wedge n}W^{(1)}(\gamma+\varepsilon_x),\\ &\quad W^{(2)}(\gamma+\varepsilon_x)\rangle_{\wedge^n(T_{\gamma+\varepsilon_x}\Gamma_X)}.\end{aligned} \tag{3.54}$$

We note that the function under the sign of integral on the right hand side of equality (3.54), considered as a function of γ and x, is integrable with respect to the measure $\mu^{(1)}(\mathrm{d}\gamma,\mathrm{d}x)$. Indeed, the integrability of the function

$$\begin{aligned}F_1(\gamma,x) := {} & \langle-\Delta_x^X W^{(1)}(\gamma+\varepsilon_x)\\ & -\langle\nabla_x^X W^{(1)}(\gamma+\varepsilon_x), \beta_\sigma(\gamma,x)\rangle_x, W^{(2)}(\gamma+\varepsilon_x)\rangle_{\wedge^n(T_{\gamma+\varepsilon_x}\Gamma_X)}\end{aligned}$$

follows from the proof of Theorem 3.5, the integrability of the function

$$F_2(\gamma,x) := -\langle(\nabla_x^X\beta_\sigma(\gamma,x))^{\wedge n}W^{(1)}(\gamma+\varepsilon_x), W^{(2)}(\gamma+\varepsilon_x)\rangle_{\wedge^n(T_{\gamma+\varepsilon_x}\Gamma_X)}$$

follows from the proof of Theorem 3.14, and the integrability of the function

$$F_3(\gamma, x) := \langle R_n(\gamma+\varepsilon_x, x)W^{(1)}(\gamma+\varepsilon_x),\, W^{(2)}(\gamma+\varepsilon_x)\rangle_{\wedge^n(T_{\gamma+\varepsilon_x}\Gamma_X)}$$

follows from the estimate

$$|F_3(\gamma, x)| \le n^2 d^4 R_\Lambda \|W^{(1)}(\gamma+\varepsilon_x)\|_{\wedge^n(T_{\gamma+\varepsilon_x}\Gamma_X)} \|W^{(2)}(\gamma+\varepsilon_x)\|_{\wedge^n(T_{\gamma+\varepsilon_x}\Gamma_X)},$$

where

$$R_\Lambda := \max_{i,j,k,l=1,\dots,d} \sup_{x\in\Lambda} |R_{i,j,k,l}(x)|,$$

$\Lambda \in \mathcal{O}_c(X)$ being such that, for some compact $\Lambda' \subset \Lambda$, $W^{(1)}(\gamma) = W^{(1)}(\gamma_{\Lambda'})$ for all $\gamma \in \Gamma_X$.

Hence, by (2.10), (3.53) and (3.54), and Theorem 3.5

$$\begin{aligned}
&\int_{\Gamma_X} \left\langle \sum_{x\in\gamma} (\boldsymbol{\delta}_{\mu,x,n}\mathbf{d}_{x,n} + \mathbf{d}_{x,n}\boldsymbol{\delta}_{\mu,x,n}) W^{(1)}(\gamma),\, W^{(2)}(\gamma) \right\rangle_{\wedge^n(T_\gamma\Gamma_X)} \mu(\mathrm{d}\gamma) \\
&\quad = \int_{\Gamma_X} \Bigg\langle \mathbf{H}^{\mathrm{B}}_{\mu,n} W^{(1)}(\gamma) + \sum_{x\in\gamma} R_n(\gamma, x) W^{(1)}(\gamma) \\
&\qquad - \sum_{x\in\gamma} (\nabla^X_x B_\mu(\gamma, x))^{\wedge n} W^{(1)}(\gamma),\, W^{(2)}(\gamma) \Bigg\rangle_{\wedge^n(T_\gamma\Gamma_X)} \mu(\mathrm{d}\gamma).
\end{aligned} \tag{3.55}$$

Next, using formulas (3.27)–(3.30), we have

$$\begin{aligned}
&(\boldsymbol{\delta}_{\mu,x,n}\mathbf{d}_{y,n} + \mathbf{d}_{y,n}\boldsymbol{\delta}_{\mu,x,n}) W^{(1)}(\gamma) \\
&\quad = -(\nabla^X_y B_\mu(\gamma, x))^{\wedge n} W^{(1)}(\gamma), \qquad \gamma \in \Gamma_X, \quad x, y \in \gamma, \quad x \neq y.
\end{aligned} \tag{3.56}$$

Thus, by (3.52), (3.55) and (3.56), we get, for μ-a.e. $\gamma \in \Gamma_X$:

$$\begin{aligned}
\mathbf{H}^{\mathrm{R}}_{\mu,n} W^{(1)}(\gamma) &= \mathbf{H}^{\mathrm{B}}_{\mu,n} W^{(1)}(\gamma, x) + R_n(\gamma) W^{(1)}(\gamma) - \sum_{x\in\gamma} (\nabla^X_x B_\mu(\gamma, x))^{\wedge n} W^{(1)}(\gamma) \\
&\quad - \sum_{x,y\in\gamma, x\neq y} (\nabla^X_y B_\mu(\gamma, x))^{\wedge n} W^{(1)}(\gamma) \\
&= \mathbf{H}^{\mathrm{B}}_{\mu,n} W^{(1)}(\gamma) + R_n(\gamma) W^{(1)}(\gamma) - (B'_\mu(\gamma))^{\wedge n} W^{(1)}(\gamma). \qquad \square
\end{aligned}$$

4. Examples

In this section, we will discuss some measures on the configuration space Γ_X to which the above results are applicable.

4.1. (Mixed) Poisson measures

Let π_z, $z > 0$, denote the Poisson measure on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ with intensity measure zm. This measure can be characterized by its Laplace transform

$$\int_\Gamma \exp[\langle f, \gamma\rangle]\pi_z(\mathrm{d}\gamma) = \exp\left(\int_X (\mathrm{e}^{f(x)} - 1)zm(\mathrm{d}x)\right), \quad f \in \mathcal{D}.$$

We refer to, e.g. [9,53] for a detailed discussion of the construction of the Poisson measure on the configuration space. The measure π_z satisfies (2.10) with $\sigma(\gamma, \mathrm{d}x) = zm(\mathrm{d}x)$, which is the so-called Mecke identity [43].

Every measure π_z is concentrated on the subset $\Xi_z \in \mathcal{B}(\Gamma_X)$ consisting of those $\gamma \in \Gamma_X$ for which

$$\lim_{n\to\infty} \frac{|\gamma_{\Lambda_n}|}{m(\Lambda_n)} = z,$$

where $(\Lambda_n)_{n=1}^\infty$ is an extending sequence of sets from $\mathcal{O}_\mathrm{c}(X)$ such that $\Lambda_n \to X$ as $n \to \infty$ (see [27,44]).

Let θ be a probability measure on $(0, \infty)$. A mixed Poisson measure π_θ is defined by

$$\pi_\theta(\cdot) := \int_0^\infty \theta(\mathrm{d}z)\pi_z(\cdot).$$

Then, evidently π_θ satisfies (2.10) with

$$\rho(\gamma, x) = zm(\mathrm{d}x) \quad \text{for } \gamma \in \Xi_z.$$

Let us suppose that

$$\int_0^\infty z^n\theta(\mathrm{d}z) < \infty \quad \text{for all } n \in \mathbb{N}.$$

Then, condition (2.8) is fulfilled, and furthermore all the theorems of Section 3 are applicable to the measure π_θ.

Let us remark the following interesting fact. The Dirichlet form on functions, $(\mathcal{E}_{\pi_\theta}, D(\mathcal{E}_{\pi_\theta}))$, is irreducible if and only if π_θ is a pure Poisson measure π_z (see [10, Theorem 6.3]). On the other hand, by Theorem 3.9, the Bochner bilinear forms $(\mathcal{E}^{\mathrm{B}}_{\pi_\theta,n}, D(\mathcal{E}^{\mathrm{B}}_{\pi_\theta,n}))$, $n \in \mathbb{N}$, are irreducible for all measures π_θ. In other words, for $\pi_\theta \neq \pi_z$ there exist square-integrable nonconstant harmonic functions, but no square-integrable Bochner-harmonic forms.

4.2. Ruelle measures

In this subsection, we will discuss a class of Gibbs measures on the configuration space over $\mathbb{R}^d$. So, let $X := \mathbb{R}^d$, $d \in \mathbb{N}$, and let $\Gamma := \Gamma_{\mathbb{R}^d}$. The volume measure m on $\mathbb{R}^d$ is now the Lebesgue measure.

A pair potential is a measurable function $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ such that $\phi(-x) = \phi(x)$. We will also suppose that $\phi(x) \in \mathbb{R}$ for $x \in \mathbb{R}^d \setminus \{0\}$. For $\Lambda \in \mathcal{O}_\mathrm{c}(\mathbb{R}^d)$, a conditional energy

$E^{\phi}_{\Lambda}: \Gamma \to \mathbb{R}\cup\{+\infty\}$ is defined by

$$E^{\phi}_{\Lambda}(\gamma) := \begin{cases} \sum_{\{x,y\}\subset\gamma,\{x,y\}\cap\Lambda\neq\emptyset}\phi(x-y) & \text{if } \sum_{\{x,y\}\subset\gamma,\{x,y\}\cap\Lambda\neq\emptyset}|\phi(x-y)|<\infty, \\ +\infty & \text{otherwise.} \end{cases}$$

Given $\Lambda\in\mathcal{O}_{\mathrm{c}}(\mathbb{R}^d)$, we define for $\gamma\in\Gamma$ and $\Delta\in\mathcal{B}(\Gamma)$

$$\begin{aligned}\Pi^{z,\phi}_{\Lambda}(\gamma,\Delta) &:= \mathbf{1}_{\{Z^{z,\phi}_{\Lambda}<\infty\}}(\gamma)\,[Z^{z,\phi}_{\Lambda}(\gamma)]^{-1} \\ &\quad\times\int_{\Gamma}\mathbf{1}_{\Delta}(\gamma_{\Lambda^c}+\gamma'_{\Lambda})\exp[-E^{\phi}_{\Lambda}(\gamma_{\Lambda^c}+\gamma'_{\Lambda})]\pi_z(\mathrm{d}\gamma'),\end{aligned}$$

where

$$Z^{z,\phi}_{\Lambda}(\gamma) := \int_{\Gamma}\exp[-E^{\phi}_{\Lambda}(\gamma_{\Lambda^c}+\gamma'_{\Lambda})]\pi_z(\mathrm{d}\gamma').$$

A probability measure μ on $(\Gamma,\mathcal{B}(\Gamma))$ is called a grand canonical Gibbs measure with interaction potential ϕ if it satisfies the Dobrushin–Lanford–Ruelle equation

$$\mu\Pi^{z,\phi}_{\Lambda} = \mu \quad \text{for all } \Lambda\in\mathcal{O}_{\mathrm{c}}(\mathbb{R}^d).$$

Let $\mathcal{G}(z,\phi)$ denote the set of all such probability measures μ.

It can be shown [26] that the unique grand canonical Gibbs measure corresponding to the free case, $\phi=0$, is the Poisson measure π_z.

We rewrite the conditional energy E^{ϕ}_{Λ} in the following form

$$E^{\phi}_{\Lambda}(\gamma) = E^{\phi}_{\Lambda}(\gamma_{\Lambda}) + W(\gamma_{\Lambda}|\gamma_{\Lambda^c}),$$

where the term

$$W(\gamma_{\Lambda}|\gamma_{\Lambda^c}) = \begin{cases} \sum_{x\in\gamma_{\Lambda},y\in\gamma_{\Lambda^c}}\phi(x-y) & \text{if } \sum_{x\in\gamma_{\Lambda},y\in\gamma_{\Lambda^c}}|\phi(x-y)|<\infty, \\ +\infty & \text{otherwise} \end{cases} \tag{4.1}$$

describes the interaction energy between γ_{Λ} and γ_{Λ^c}. Analogously, we define $W(\gamma'|\gamma'')$ when $\gamma'\cap\gamma''=\emptyset$.

We suppose that the interaction potential ϕ is stable, i.e., the following condition is satisfied:

(S) (Stability) There exists $B\geq 0$ such that, for any $\Lambda\in\mathcal{O}_{\mathrm{c}}(\mathbb{R}^d)$ and for all $\gamma\in\Gamma_{\Lambda}$:

$$E^{\phi}_{\Lambda}(\gamma)\geq -B|\gamma|.$$

(Notice that the stability condition automatically implies that the potential ϕ is semi-bounded from below.)

Then, any $\mu\in\mathcal{G}(z,\phi)$ satisfies identity (2.10) with

$$\rho(\gamma,x) = z\exp[-W(\{x\}|\gamma)]. \tag{4.2}$$

In fact, this property uniquely characterizes a Gibbs measure in the sense that any probability measure μ on $(\Gamma,\mathcal{B}(\Gamma))$ belongs to $\mathcal{G}(z,\phi)$ if and only if μ satisfies (2.10) with $\rho(\gamma,x)$ given by (4.2) (cf. [45], see also [32]).

Let us now describe a class of Gibbs measures which appears in classical statistical mechanics of continuous systems [51]. For every $r = (r^1, \dots, r^d) \in \mathbb{Z}^d$, we define a cube

$$Q_r := \{x \in \mathbb{R}^d | r^i - \tfrac{1}{2} \le x^i < r^i + \tfrac{1}{2}\}.$$

These cubes form a partition of $\mathbb{R}^d$. For any $\gamma \in \Gamma$, we set $\gamma_r := \gamma_{Q_r}$, $r \in \mathbb{Z}^d$. For $N \in \mathbb{N}$ let Λ_N be the cube with side length $2N-1$, centered at the origin in $\mathbb{R}^d$. Λ_N is then a union of $(2N-1)^d$ unit cubes of the form Q_r.

We formulate the following conditions on the interaction.

(SS) (Superstability) There exist $A > 0$, $B \ge 0$ such that if $\gamma \in \Gamma_{\Lambda_N}$ for some N, then

$$E^{\phi}_{\Lambda_N}(\gamma) \ge \sum_{r \in \mathbb{Z}^d} [A|\gamma_r|^2 - B|\gamma_r|].$$

This condition is evidently stronger than (S).

(LR) (Lower regularity) There exists a decreasing positive function $a : \mathbb{N} \to \mathbb{R}_+$ such that

$$\sum_{r \in \mathbb{Z}^d} a(\|r\|) < \infty$$

and for any Λ', Λ'' which are finite unions of cubes Q_r and disjoint, with $\gamma' \in \Gamma_{\Lambda'}$, $\gamma'' \in \Gamma_{\Lambda''}$:

$$W(\gamma'|\gamma'') \ge - \sum_{r', r'' \in \mathbb{Z}^d} a(\|r' - r''\|)|\gamma'_{r'}|\,|\gamma''_{r''}|.$$

Here, $\|\cdot\|$ denotes the maximum norm on $\mathbb{R}^d$.

(I) (Integrability) We have

$$\int_{\mathbb{R}^d} |1 - \mathrm{e}^{-\phi(x)}|\, m(\mathrm{d}x) < +\infty.$$

A probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ is called tempered if μ is supported by

$$S_\infty := \bigcup_{n=1}^{\infty} S_n,$$

where

$$S_n := \left\{ \gamma \in \Gamma | \forall\, N \in \mathbb{N} \sum_{r \in \Lambda_N \cap \mathbb{Z}^d} |\gamma_r|^2 \le n^2 |\Lambda_N \cap \mathcal{Z}^d| \right\}.$$

By $\mathcal{G}^t(z, \phi) \subset \mathcal{G}(z, \phi)$ we denote the set of all tempered grand canonical Gibbs measures (Ruelle measures for short). Due to [51] the set $\mathcal{G}^t(z, \phi)$ is nonempty for all $z > 0$ and any potential ϕ satisfying conditions (SS), (LR), and (I).

Let us now recall the so-called Ruelle bound (cf. [51]).

Theorem 4.1. *Let ϕ be a pair potential satisfying conditions* (SS), (LR), *and* (I), *and let $\mu \in \mathcal{G}^t(z,\phi)$, $z > 0$. Then, for any $n \in \mathbb{N}$ and any measurable symmetric function $f^{(n)} : (\mathbb{R}^d)^n \to [0,\infty]$, we have*

$$\int_\Gamma \sum_{\{x_1,\dots,x_n\}\subset\gamma} f^{(n)}(x_1,\dots,x_n)\mu(\mathrm{d}\gamma) = \frac{1}{n!}\int_{(\mathbb{R}^d)^n} f^{(n)}(x_1,\dots,x_n)k_\mu^{(n)}(x_1,\dots,x_n)m(\mathrm{d}x_1)\cdots m(\mathrm{d}x_n),$$

where $k_\mu^{(n)}$ is a nonnegative measurable symmetric function on $(\mathbb{R}^d)^n$, called the nth correlation function of the measure μ, and this function satisfies the following estimate

$$\forall\,(x_1,\dots,x_n) \in (\mathbb{R}^d)^n : k_\mu^{(n)}(x_1,\dots,x_n) \le \xi^n, \tag{4.3}$$

where $\xi > 0$ is independent of n.

The above theorem particularly implies that any Ruelle measure μ satisfies (2.8).

We suppose:

(S1) There exists $r > 0$ such that

$$\int_{B(r)^c} |\phi(x)|m(\mathrm{d}x) < \infty,$$

where $B(r)$ denotes the open ball in $\mathbb{R}^d$ of radius r centered at the origin.

Lemma 4.2. *Let* (SS), (LR), (I), *and* (S1) *hold. Then*:

$$\sum_{y\in\gamma} |\phi(x-y)| < \infty \quad \text{for } \mu\otimes m\text{-a.e. } (\gamma,x) \in \Gamma\times\mathbb{R}.$$

Moreover, for $\mu\otimes m$-a.e. $(\gamma,x) \in \Gamma\times\mathbb{R}^d$

$$\rho(\gamma,x) = z\exp\left[-\sum_{y\in\gamma}\phi(x-y)\right] > 0.$$

Proof. It is enough to show that, for any $\Lambda \in \mathcal{O}_{\mathrm{c}}(\mathbb{R})$

$$\sum_{y\in\gamma_{(\Lambda^r)^c}} |\phi(x-y)| < \infty \quad \text{for } \mu\otimes m\text{-a.e. } (\gamma,x) \in \Gamma\times\Lambda, \tag{4.4}$$

where $\Lambda^r := \{y \in \mathbb{R}^d : d(y,\Lambda) \le r\}$, $d(y,\Lambda)$ denoting the distance from y to Λ.

By Theorem 4.1 and (S1):

$$\int_\Gamma \mu(\mathrm{d}\gamma)\int_\Lambda m(\mathrm{d}x)\sum_{y\in\gamma_{(\Lambda^r)^c}}|\phi(x-y)|$$
$$=\int_\Lambda m(\mathrm{d}x)\int_\Gamma \mu(\mathrm{d}\gamma)\int_{\mathbb{R}^d}\gamma(\mathrm{d}y)|\phi(x-y)|\mathbf{1}_{(\Lambda^r)^c}(y)$$
$$=\int_\Lambda m(dx)\int_{\mathbb{R}^d} m(\mathrm{d}y)k_\mu^{(1)}(y)|\phi(x-y)|\mathbf{1}_{(\Lambda^r)^c}(y)$$
$$\le \xi\int_\Lambda m(\mathrm{d}x)\int_{(\Lambda^r)^c} m(\mathrm{d}y)|\phi(x-y)|$$
$$\le \xi m(\Lambda)\int_{B(r)^c}|\phi(y)|m(\mathrm{d}y)<\infty,$$

which implies (4.4). The second conclusion of the lemma now trivially follows from (4.1) and (4.2). □

We also suppose that the two following conditions are satisfied (compare with [10]).

(D) (Differentiability) $\mathrm{e}^{-\phi}$ is weakly differentiable on $\mathbb{R}^d$, ϕ is weakly differentiable on $\mathbb{R}^d\setminus\{0\}$, and the weak gradient $\nabla\phi$ (which is a locally m-integrable function on $\mathbb{R}^d\setminus\{0\}$) considered as an m-a.e. defined function on $\mathbb{R}^d$ satisfies

$$\nabla\phi\in L^1(\mathbb{R}^d,\ \mathrm{e}^{-\phi}m)\cap L^3(\mathbb{R}^d,\ \mathrm{e}^{-\phi}m). \tag{4.5}$$

Remark 4.3. It follows from (D) that

$$\nabla\,\mathrm{e}^{-\phi}=-\nabla\phi\,\mathrm{e}^{-\phi}\quad m\text{-a.e. on }\mathbb{R}^d.$$

(S2) There exists $R>0$ such that

$$\int_{B(R)^c}|\nabla\phi(x)|\,m(dx)<\infty.$$

Proposition 4.4. *Let* (SS), (LR), (I), (D), (S1) *and* (S2) *hold. Then, any* $\mu\in\mathcal{G}^t(z,\phi)$, $z>0$, *satisfies the conditions of* Theorem 3.5 *and*

$$B_\mu(\gamma,x)=-\sum_{y\in\gamma\setminus\{x\}}\nabla\phi(x-y),\quad x\in\gamma,\ \mu\text{-a.e. }\gamma\in\Gamma_X. \tag{4.6}$$

Proof. We first prove that, for μ-a.e. $\gamma\in\Gamma_X$ $\rho(\gamma,\cdot)$ is weakly differentiable on $\mathbb{R}^d$. We fix any $f\in\mathcal{D}$ and v, a smooth vector field on $\mathbb{R}^d$ with compact support, and let $\Lambda\in\mathcal{O}_c(\mathbb{R}^d)$ be such that the supports of both f and v are contained in Λ. Let $(\Lambda_N)_{N=1}^\infty$ be the sequence of subsets of $\mathbb{R}^d$ as in (SS). Let $N\in\mathbb{N}$ be so big that $\Lambda^R\subset\Lambda_N$. Then, using Remark 4.3, we get

$$\begin{aligned}
&\int_\Lambda \exp\left[-\sum_{y\in\gamma_{\Lambda_N}}\phi(x-y)\right]\langle\nabla f(x), v(x)\rangle zm(\mathrm{d}x)\\
&\quad= \int_\Lambda \exp\left[-\sum_{y\in\gamma_{\Lambda_N}}\phi(x-y)\right] f(x)\left(\sum_{y\in\gamma_{\Lambda_N}}\langle\nabla\phi(x-y), v(x)\rangle - \operatorname{div} v(x)\right) zm(\mathrm{d}x)\\
&\quad= \int_\Lambda \exp\left[-\sum_{y\in\gamma_{\Lambda_N}}\phi(x-y)\right] f(x)\left(\sum_{y\in\gamma_{\Lambda^R}}\langle\nabla\phi(x-y), v(x)\rangle - \operatorname{div} v(x)\right) zm(\mathrm{d}x)\\
&\qquad+ \int_\Lambda \exp\left[-\sum_{y\in\gamma_{\Lambda_N}}\phi(x-y)\right] f(x)\left(\sum_{y\in\gamma_{\Lambda_N\setminus\Lambda^R}}\langle\nabla\phi(x-y), v(x)\rangle\right) zm(\mathrm{d}x).
\end{aligned}\tag{4.7}$$

We know from [52, Lemma 5.1, Proposition 5.2 and its proof] that, for each $\gamma\in S_\infty$, there exists a constant $C(\gamma)>0$ such that

$$\forall\, N\in\mathbb{N}, \forall\, x\in\Lambda : \exp[-W(\{x\}|\gamma_{\Lambda_N})]\le C(\gamma). \tag{4.8}$$

Moreover, analogously to the proof of (4.4), we conclude from (S2) that

$$\int_\Lambda \sum_{y\in\gamma_{(\Lambda^R)^c}} |\nabla\phi(x-y)|zm(\mathrm{d}x)<\infty \quad \text{for } \mu\text{-a.e. } \gamma\in\Gamma_X. \tag{4.9}$$

Now, by virtue of Lemma 4.2, (4.5), (4.7)–(4.9), and the majorized convergence theorem, we get

$$\begin{aligned}
&\int_\Lambda \exp\left[-\sum_{y\in\gamma}\phi(x-y)\right]\langle\nabla f(x), v(x)\rangle zm(\mathrm{d}x)\\
&\quad= \int_\Lambda \exp\left[-\sum_{y\in\gamma}\phi(x-y)\right] f(x)\left(\sum_{y\in\gamma}\langle\nabla\phi(x-y), v(x)\rangle - \operatorname{div} v(x)\right) zm(\mathrm{d}x).
\end{aligned}$$

Therefore, for μ-a.e. $\gamma\in\Gamma$, $\rho(\gamma,\cdot)$ is weakly differentiable on $\mathbb{R}^d$ and

$$\beta_\sigma(\gamma,x) = -\sum_{y\in\gamma}\nabla\phi(x-y),$$

so that B_μ is given by (4.6).

Finally, let us show that, for any $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$:

$$\int_\Gamma \left(\sum_{x\in\gamma_\Lambda} \sum_{y\in\gamma\setminus\{x\}} |\nabla\phi(x-y)| \right)^3 \mu(\mathrm{d}\gamma)$$
$$= \frac{1}{8} \int_\Gamma \left(\sum_{\{x,y\}\subset\gamma} |\nabla\phi(x-y)|(\mathbf{1}_\Lambda(x)+\mathbf{1}_\Lambda(y)) \right)^3 \mu(\mathrm{d}\gamma) < \infty, \tag{4.10}$$

which implies (3.8) with $\varepsilon = 1$.

The proof of (4.10) is essentially analogous to that of [10, Lemma 4.1], so we only sketch it. By using [32, Proposition 3.11] and Theorem 4.1, we get, for any nonnegative symmetric function $\varphi^{(2)}(x, y)$ on $(\mathbb{R}^d)^2$:

$$\int_\Gamma \left(\sum_{\{x,y\}\subset\gamma} \varphi^{(2)}(x, y) \right)^3 \mu(\mathrm{d}\gamma)$$
$$= c_1 \int_{(\mathbb{R}^d)^6} \varphi^{(2)}(x_1, x_2)\varphi^{(2)}(x_3, x_4)\varphi^{(2)}(x_5, x_6)k_\mu^{(6)}(x_1, \dots, x_6)m(\mathrm{d}x_1)\cdots m(\mathrm{d}x_6)$$
$$+ c_2 \int_{(\mathbb{R}^d)^5} \varphi^{(2)}(x_1, x_2)\varphi^{(2)}(x_1, x_3)\varphi^{(2)}(x_4, x_5)k_\mu^{(5)}(x_1, \dots, x_5)m(\mathrm{d}x_1)\cdots m(\mathrm{d}x_5)$$
$$+ \int_{(\mathbb{R}^d)^4} (c_3\varphi^{(2)}(x_1, x_2)^2\varphi^{(2)}(x_3, x_4) + c_4\varphi^{(2)}(x_1, x_2)\varphi^{(2)}(x_2, x_3)\varphi^{(2)}(x_3, x_4)$$
$$+c_5\varphi^{(2)}(x_1, x_2)\varphi^{(2)}(x_1, x_3)\varphi^{(2)}(x_1, x_4))k_\mu^{(4)}(x_1, \dots, x_4)m(\mathrm{d}x_1)\cdots m(\mathrm{d}x_4)$$
$$+ c_6 \int_{(\mathbb{R}^d)^3} (\varphi^{(2)}(x_1, x_2)^2\varphi^{(2)}(x_1, x_3) + c_7\varphi^{(2)}(x_1, x_2)\varphi^{(2)}(x_1, x_3)\varphi^{(2)}(x_2, x_3))$$
$$\times k_\mu^{(3)}(x_1, x_2, x_3)m(\mathrm{d}x_1)m(\mathrm{d}x_2)m(\mathrm{d}x_3)$$
$$+c_8 \int_{(\mathbb{R}^d)^2} \varphi^{(2)}(x_1, x_2)^3k_\mu^{(2)}(x_1, x_2)m(\mathrm{d}x_1)m(\mathrm{d}x_2), \tag{4.11}$$

where $c_1, \dots, c_8 > 0$. We recall also the estimate (cf. [10, formula (4.29)])

$$\forall\,(x_1, \dots, x_n) \in (\mathbb{R}^d)^n : k_\mu^{(n)}(x_1, \dots, x_n) \le R_n \exp\left[-\sum_{1\le i<j\le n} \phi(x_i - x_j) \right], \tag{4.12}$$

where $n \in \mathbb{N}$ and $R_n > 0$. Finally, one proves (4.10) by using (4.5), (4.11) and (4.12), and the semi-boundedness of the potential ϕ from below. □

Proposition 4.5. *Let the conditions of Proposition 4.4 be satisfied, let for some $\mathcal{R} > 0$*

$$\phi(x) \le 0, \quad x \in B(\mathcal{R})^c \tag{4.13}$$

and let one of the two following conditions is satisfied:

(a) $\phi \in C(\mathbb{R}^d)$ *and for each* $\gamma \in S_\infty$ *the series* $\sum_{x\in\gamma} \phi(\cdot - x)$ *converges locally uniformly on* X.
(b) $d \geq 2$, $\phi \in C(\mathbb{R}^d \setminus \{0\})$, *and for each* $\gamma \in S_\infty$ *the series* $\sum_{x\in\gamma} \phi(\cdot - x)$ *converges locally uniformly on* $X \setminus \gamma$.

Then, the conditions of Theorem 3.9 are satisfied for each $\mu \in \mathcal{G}^t(z, \phi)$.

Proof. Evidently, (a) implies condition (i) of Theorem 3.9 and (b) does (ii), so that we only have to show (3.15). Let us fix any $\gamma \in S_\infty$. It follows from the definition of S_∞ that there exists $C = C(\gamma) \in \mathbb{N}$ such that

$$|\gamma_{\Lambda_N}| \leq Cm(\Lambda_N), \qquad N \in \mathbb{N}. \tag{4.14}$$

Let us assume that in (4.13) $\mathcal{R} = 1/4$, otherwise only a trivial modification of the proof is needed.

For $a > 0$, let $[a]$ denote the integer part of a. Supposing that there exist $[1/2(2N-1)^d]+1$ Q_r cubes in Λ_N which contain at least $3C$ points of γ, we come to a contradiction with (4.14). Therefore, there exist at least $(2N-1)^d - [1/2(2N-1)^d]$ cubes which contain less than $3C$ points of γ. Setting $N \to \infty$, we conclude that there exists an infinite sequence $\{Q_{r(k)},\ k \in \mathbb{N}\}$ of cubes which contain $< 3C$ points of γ. Let x_k denote the center of the cube $Q_{r(k)}$. Then:

$$\forall\, x \in B(x_k, \tfrac{1}{4}), k \in \mathbb{N}: \quad |B(x, \tfrac{1}{4}) \cap \gamma| < 3C. \tag{4.15}$$

In case of (a), we get by (4.15):

$$\forall\, x \in B(x_k, \tfrac{1}{4}), k \in \mathbb{N}: \quad \sum_{y\in\gamma} \phi(x - y) \leq \text{const}. \tag{4.16}$$

and hence

$$\forall\, x \in B(x_k, \tfrac{1}{4}), k \in \mathbb{N}: \qquad \rho(\gamma, x) \geq \exp(-\text{const}). \tag{4.17}$$

Therefore, $\sigma(\gamma, \cdot)$, as well as all measures $\sigma^{(k)}(\gamma, \cdot)$, $k \geq 2$, are infinite measures.

In the case of (b), we proceed as follows. Any ball $B(x_k, 1/4)$ contains $3C$ open disjoint balls of of radius $1/(12C)$, and at least one of these balls does not contain any point of γ. Therefore, each $B(x_k, 1/4)$ contains a ball $B(y_k, 1/(24C))$ such that

$$\forall\, x \in B\left(y_k, \frac{1}{24C}\right): \qquad \inf_{y\in\gamma}|x - y| \geq \frac{1}{24C}. \tag{4.18}$$

By (b) the function ϕ is bounded on $\{x \in \mathbb{R}^d : 1/(24C) \leq |x| \leq \mathcal{R}\}$, and therefore by (4.15) and (4.18), we again conclude that all $\sigma^{(k)}(\gamma, \cdot)$, $k \in \mathbb{N}$ are infinite measures. □

Proposition 4.6. *Let* (SS), (LR), (I), *and* (S2) *hold. Furthermore, let the interaction potential* ϕ *satisfy the following conditions*:

(i) $\phi \in C^2(\mathbb{R}^d \setminus \{0\})$, $\mathrm{e}^{-\phi}$ *is continuous on* $\mathbb{R}^d$, *and* $\mathrm{e}^{-\phi}\nabla\phi$ *extends to a continuous vector-valued function on* $\mathbb{R}^d$.

(ii) *For each* $\gamma \in S_\infty$, *the series* $\sum_{x\in\gamma}\phi(\cdot - x)$, $\sum_{x\in\gamma}\nabla\phi(\cdot - x)$, *and* $\sum_{x\in\gamma}\phi''(\cdot - x)$ *converge locally uniformly on* $X \setminus \gamma$.

(iii) (4.5) *holds, and furthermore*:

$$\phi'' \in L^1(\mathbb{R}^d, \mathrm{e}^{-\phi}m) \cap L^3(\mathbb{R}^d, \mathrm{e}^{-\phi}m). \tag{4.19}$$

Then, any $\mu \in \mathcal{G}^t(z,\phi)$, $z > 0$, *satisfies the conditions of* Theorem 3.14

Proof. As easily seen, conditions (i)–(iii) of Theorem 3.14 are now satisfied. Indeed, let us fix any $\gamma \in S_\infty$. By condition (ii):

$$\rho(\gamma, x) = \exp\left[-\sum_{y\in\gamma}\phi(x-y)\right] > 0, \quad x \in \mathbb{R}^d \setminus \gamma. \tag{4.20}$$

It follows from the definition of S_∞ that, for any $y \in S_\infty$, $\gamma \setminus \{y\}$ again belongs to S_∞, and therefore, the function

$$\mathcal{O}_{\gamma,y} \ni x \mapsto \exp\left[-\sum_{z\in\gamma}\phi(x-z)\right] = \exp[-\phi(x-y)]\exp\left[-\sum_{z\in\gamma\setminus\{y\}}\phi(x-z)\right]$$

is continuous by (i) and (ii). Hence, $\rho(\gamma,\cdot)$ is continuous on $\mathbb{R}^d$. Moreover, by (i), (ii), and (4.20), the function $\rho(\gamma,\cdot)$ is two times differentiable on $\mathbb{R}^d \setminus \gamma$, and analogously to the above, we conclude that the form

$$\begin{aligned}\mathcal{O}_{\gamma,y} \ni x \mapsto \nabla_x\rho(\gamma, x) = -\exp&\left[-\phi(x-y) - \sum_{z\in\gamma\setminus\{y\}}\phi(x-z)\right]\\ &\times\left(\nabla\phi(x-y) + \sum_{z\in\gamma\setminus\{y\}}\nabla\phi(x-z)\right)\end{aligned}$$

is continuous on $\mathcal{O}_{\gamma,y}$, so that $\nabla_x\rho(\gamma,\cdot)$ is continuous on $\mathbb{R}^d$. Finally, for any $x \in \mathbb{R}^d \setminus \gamma$:

$$\begin{aligned}\nabla_x\rho(\gamma+\varepsilon_y, x) = -\exp&\left[-\phi(x-y) - \sum_{z\in\gamma}\phi(x-z)\right]\\ &\times\left(\nabla\phi(x-y) + \sum_{z\in\gamma}\nabla\phi(x-z)\right)\end{aligned}$$

is differentiable in y on $\mathbb{R}^d \setminus (\gamma \cup \{x\})$, and

$$\begin{aligned}&\frac{\rho(\gamma+\varepsilon_x, y)}{\rho(\gamma+\varepsilon_y, x)}\nabla_x\rho(\gamma+\varepsilon_y, x)\\ &\quad = -\exp\left[-\phi(x-y) - \sum_{z\in\gamma}\phi(z-y)\right]\left(\nabla\phi(x-y) + \sum_{z\in\gamma}\nabla\phi(x-z)\right)\end{aligned}$$

extends to a continuous form in y on $\mathbb{R}^d$.

That (3.8) holds follows from (4.5) and (S2) (see the proof of Proposition 4.4). Thus, it only remains to show that (3.40) is also satisfied.

It follows from the above that, for each $\gamma \in S_\infty$:

$$B_\mu(\gamma, x) = -\sum_{y\in\gamma\setminus\{x\}} \nabla\phi(x-y), \quad x \in \gamma,$$

and hence, by (i), (ii), we get for any $x, y \in \gamma$:

$$\nabla_y B_\mu(\gamma, x) = \begin{cases} \phi''(x-y) & \text{if } x \neq y, \\ -\sum_{z\in\gamma\setminus\{x\}}\phi''(x-z) & \text{if } x = y. \end{cases}$$

Hence, for any $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$, we get

$$\begin{aligned} &\int_\Gamma \left(\sum_{y\in\gamma}\sum_{x\in\gamma_\Lambda} \|\nabla_y B_\mu(\gamma, x)\|\right)^3 \mu(\mathrm{d}\gamma) \\ &\quad = \int_\Gamma \left(\sum_{x\in\gamma_\Lambda} \|\nabla_x B_\mu(\gamma, x)\| + \sum_{x\in\gamma_\Lambda}\sum_{y\in\gamma\setminus\{x\}} \|\nabla_y B_\mu(\gamma, x)\|\right)^3 \mu(\mathrm{d}\gamma) \\ &\quad \leq \left(2\sum_{x\in\gamma_\Lambda}\sum_{y\in\gamma\setminus\{x\}} \|\phi''(x-y)\|\right)^3 \mu(\mathrm{d}\gamma). \end{aligned} \tag{4.21}$$

The finiteness of the latter integral in (4.21) follows from (4.19) in the same way as (4.10) follows from (4.5). □

Remark 4.7. Let the interaction potential ϕ satisfy conditions of Proposition 4.6. Then, by using Lemma 2.1, Theorem 3.16, and Proposition 4.4, we easily see that, for every $W \in \mathcal{D}\Omega^1$:

$$\begin{aligned} \mathcal{E}^{\mathrm{R}}_{\mu,1}(W, W) &= \int_\Gamma \mu(\mathrm{d}\gamma)\int_{\mathbb{R}^d} m(\mathrm{d}x)\exp\left(-\sum_{y\in\gamma}\phi(x-y)\right)|\nabla_x W(\gamma+\varepsilon_x, x)|^2 \\ &\quad + \frac{1}{2}\int \mu(\mathrm{d}\gamma)\int_{\mathbb{R}^d} m(\mathrm{d}x)\int_{\mathbb{R}^d} m(\mathrm{d}y) \\ &\quad \times \exp\left(-\sum_{x'\in\gamma}\phi(x-x') - \sum_{y'\in\gamma}\phi(y-y') - \phi(x-y)\right) \\ &\quad \times \phi''(x-y)(W(\gamma+\varepsilon_x+\varepsilon_y, x) - W(\gamma+\varepsilon_x+\varepsilon_y, y)) \\ &\quad \times (W(\gamma+\varepsilon_x+\varepsilon_y, x) - W(\gamma+\varepsilon_x+\varepsilon_y, y)). \end{aligned}$$

Finally, we present several examples of potentials which satisfy conditions of Propositions 4.4 and 4.6.

Example 1. $\phi \in C_0^2(\mathbb{R}^d)$, $\phi \geq 0$ on $\mathbb{R}^d$, and $\phi(0) > 0$.

Example 2. (Lennard–Jones type potentials) $\phi \in C^2(\mathbb{R}^d \setminus \{0\})$, $\phi \geq 0$ on $\mathbb{R}^d$, $\phi(x) = c|x|^{-\alpha}$ for $x \in B(r_1)$, $\phi(x) = 0$ for $x \in B(r_2)^c$, where $c > 0$, $\alpha > 0$, $0 < r_1 < r_2 < \infty$.

Example 3. (Lennard–Jones 6–12 potentials) $d = 3$, $\phi(x) = c(|x|^{-12} - |x|^{-6})$, $c > 0$.

4.3. Gibbs measures on configuration spaces over manifolds

In this subsection, we will shortly discuss the case of a Gibbs measure μ on Γ_X, where X is again a general manifold.

We formulate the following conditions on the interaction potential ϕ, which is now a symmetric functions $\phi : X^2 \to \mathbb{R} \cup \{+\infty\}$.

(S) (Stability) There exists $B \geq 0$ such that, for any $\Lambda \in \mathcal{O}_c(X)$ and for all $\gamma \in \Gamma_\Lambda$:

$$E_\Lambda^\phi(\gamma) := \sum_{\{x,y\}\subset\gamma} \phi(x, y) \geq -B|\gamma|.$$

(I) (Integrability) We have

$$C := \operatorname*{ess\,sup}_{x\in X} \int_X |\mathrm{e}^{-\phi(x,y)} - 1| m(\mathrm{d}y) < \infty.$$

(F) (Finite range) There exists $R > 0$ such that

$$\phi(x, y) = 0 \quad \text{if } \mathrm{d}(x, y) \geq R.$$

In a completely analogous way as for the case of $\mathbb{R}^d$, one defines a Gibbs measure μ corresponding to the interaction potential ϕ and activity parameter $z > 0$, and one denotes by $\mathcal{G}(z, \phi)$ the set of all such measures.

Theorem 4.8 ([33–35]).

(1) *Let* (S), (I), *and* (F) *hold, and let* $z > 0$ *be such that*

$$z < \frac{1}{2\,\mathrm{e}}(\mathrm{e}^{2B}C)^{-1},$$

where B *and* C *are as in* (S) *and* (I), *respectively. Then, there exists a Gibbs measure* $\mu \in \mathcal{G}(z, \phi)$ *such that the correlation functions* $k_\mu^{(n)}$ *of the measure* μ *satisfy the Ruelle bound* (4.3).

(2) *Let* ϕ *be a nonnegative potential which fulfills* (I) *and* (F). *Then, for each* $z > 0$, *there exists a Gibbs measure* $\mu \in \mathcal{G}(z, \phi)$ *such that the correlation functions* $k_\mu^{(n)}$ *of the measure* μ *satisfy the Ruelle bound* (4.3).

Proposition 4.9. *Suppose the conditions of* Theorem 4.8 *are satisfied and furthermore suppose that the interaction potential* ϕ *satisfies the following conditions*:

(i) $\phi \in C^2(X^2 \setminus \tilde{X}^2)$, $e^{-\phi}$ *is continuous on* X^2, *and* $e^{-\phi}\nabla_1^X\phi$ *extends to a continuous vector field on* X^2 (*here* $\nabla_1^X\phi$ *denotes the gradient of the function* ϕ *in the first variable*);

(ii) *We have*

$$\operatorname*{ess\,sup}_{x\in X} \int_X |(\nabla_x^X)^k \phi(x,y)|^n \exp(-\phi(x,y)) m(\mathrm{d}y) < \infty, \qquad k=1,2, \quad n=1,2,3.$$

Let $\mu \in \mathcal{G}(z,\phi)$ *be as in* Theorem 4.8. *Then,* μ *satisfies the conditions of* Theorems 3.5 and 3.14.

Proof. The proof of this proposition essentially follows the lines of the proof of Proposition 4.6, and is even easier, since due to condition (F) all series $\sum_{y\in\gamma}\phi(x,y)$, $\gamma \in \Gamma_X$, $x \in X\setminus\gamma$, are finite. □

Proposition 4.10. *Suppose that the manifold X satisfies the following condition*:

$$\forall r>0: \quad 0 < \inf_{x\in X} m(B(x,r)) \le \sup_{x\in X} m(B(x,r)) < \infty. \tag{4.22}$$

Assume that the conditions of Proposition 4.9 *are satisfied and either* ϕ *is a continuous bounded function on* X^2, *or* $d \ge 2$ *and*

$$\forall r>0: \quad \sup_{x\in X} \sup_{y\in X, \mathrm{d}(x,y)\ge r} |\phi(x,y)| < \infty.$$

Let $\mu \in \mathcal{G}(z,\phi)$ *be as in* Theorem 4.8. *Then, the conditions of* Theorem 3.9 *are satisfied.*

Remark 4.11. Condition (4.22) is satisfied in the case of a manifold having bounded geometry (see [21]). The upper estimate $\sup_{x\in X} m(B(x,r)) < \infty$, $r>0$, holds for manifolds having nonnegative Ricci curvature (see, e.g. [21, Proposition 5.5.1]).

Proof. Let us fix any sequence $\{B(x_n, 2R), n \in \mathbb{N}\}$ of disjoint balls in X, where R is as in (F). Let

$$\Lambda_N := \bigcup_{n=1}^N B(x_n, 2R), \qquad N \in \mathbb{N}. \tag{4.23}$$

By (4.22), $m(\Lambda_N) \to \infty$ as $N \to \infty$. By Theorem 4.8, the correlation functions $k_\mu^{(n)}$ satisfy the Ruelle bound. Hence, it follows from (the proof of) [32, Theorem 2.5.4] that there exists a subsequence $\{\Lambda_{N(k)}, k \in \mathbb{N}\}$ such that, for μ-a.e. $\gamma \in \Gamma$, there exists $C = C(\gamma) > 0$ satisfying

$$|\gamma_{\Lambda_{N(k)}}| \le C m(\Lambda_{N(k)}) \quad \text{for all } k \in \mathbb{N}. \tag{4.24}$$

By (4.22)–(4.24):

$$|\gamma_{\Lambda_{N(k)}}| \le C \left(\sup_{x\in X} m(B(x, 2R)) \right) N(k), \qquad k \in \mathbb{N}.$$

Since by (4.22) $\inf_{x\in X} m(B(x,r)) > 0$, $r > 0$, the rest of the proof is now completely analogous to the proof of Proposition 4.5. □

Example. Suppose that the manifold X satisfies (4.22), and for some $R > 0$

$$\sup_{x\in X}\sup_{y\in B(x,R)} |\nabla_y^X f(x,y)|_{T_yX} < \infty, \quad k = 1, 2,$$

where

$$X^2 \ni (x,y) \mapsto f(x,y) := \mathrm{d}(x,y)^2 \in \mathbb{R}.$$

(For example, these conditions are satisfied if the manifold has a periodical structure.) Let $\Phi \in C^2([0,\infty))$ be such that $\Phi \geq 0$ on $[0,\infty)$ and $\Phi(x) = 0$ for $x \geq R^2$. Then, the potential $\phi(x,y) := \Phi(f(x,y))$ satisfies the conditions of Propositions 4.9 and 4.10.

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